Bounds on Generalized Symmetric Numerical Semigroups

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Abstract

Numerical semigroups are combinatorial objects that are easy to define, but have rich connections to other fields. Certain families of numerical semigroups are of particular interest because of their connections to algebraic geometry. We focus on one such family known as symmetric semigroups, and analyze the rate of growth of the number of symmetric semigroups $S(g)$ with genus g. Then, we partition semigroups of genus g by their Frobenius number, and denote by $N(g, F)$ the number of semigroups with genus g and Frobenius number F. We extend results from $S(g)$ to $N(g, 2g-k)$ for k fixed in the range $1 \leq k \leq g$. We state a conjecture about the local behavior of the ratio $\frac{S(g+1)}{S(g)}$, depending on the residue of $g \pmod 3$. Finally, we generalize this conjecture to include $N(g, 2g-k)$ for fixed k.

1 Introduction

Let Γ be a subset of the non-negative integers $\mathbb{N}_0$. We say that Γ is a *numerical semigroup* if it is closed under addition, contains the additive identity 0, and has finite complement $\mathbb{N}_0 \setminus \Gamma$. The integers in $\mathbb{N}_0 \setminus \Gamma$ are called *gaps*, and the number of gaps is called the *genus* of Γ, denoted by g or $g(\Gamma)$. Finally, we define the largest gap to be the *Frobenius number* of the numerical semigroup, denoted $F(\Gamma)$.

The number of numerical semigroups with genus g, which we denote $N(g)$, has been explored in [2], [3], and [8], and a summary of known results is explained in [4]. In [3], $N(g)$ is bounded by $2F_g \leq N(g) \leq 3 \cdot 2^{g-3}$ for all $g \geq 3$, where F_g denotes the g^{th} Fibonacci number. In [8], it is shown that $\lim_{g\to\infty} \frac{N(g)}{\phi^g} = C$, where C is some constant greater than 3.78 and $\phi = \frac{1+\sqrt{5}}{2}$. In [2], it is conjectured that $N(g) \geq N(g-1) + N(g-2)$ for $g \geq 2$. However, the much weaker statement $N(g) \geq N(g-1)$ has not yet been proven. The simply-stated definitions that describe numerical semigroups often conceal complex proofs, and many problems in the area are still unanswered.

Numerical semigroups have become a topic of interest over the last few decades because of their connections to algebraic geometry. For example, given a field F, the valuations of the elements of the ring of formal power series $F[[t^{n_1}, t^{n_2}, \ldots, t^{n_e}]]$ are precisely to the elements of $\langle n_1, n_2, \ldots, n_e \rangle$ [6]. Because of this link, some invariants—such as the *embedding dimension* for the cardinality of the minimal generators, or the *conductor* for the smallest element greater than every gap—borrow terms from algebraic geometry [5, 6].

Certain families of numerical semigroups are also of interest because of their connection to algebraic geometry. We explore one such family in Section 3, where we focus on the number of *symmetric semigroups* $S(g)$ with genus g. We begin by bounding $S(g)$ with respect to the total number of semigroups $N(g)$.

Lemma 1.1. *For all $g \geq 2$, the inequality $S(g) \leq N(g-2)+1$ holds.*

We then prove the following stronger result.

Theorem 1.2. *We have*

$$\lim_{g\to\infty} \frac{S(g)}{N(g)} = 0.$$

In Section 4, we define a partition of $N(g)$ according to Frobenius number, and denote by $N(g, F)$ the number of semigroups with genus g and Frobenius number F. We then rewrite $F = 2g - k$ for k in the range $1 \leq k \leq g$ such that the base case $N(g, 2g-1) = S(g)$. Using this definition, we generalize our results from $S(g)$ to every subset $N(g, 2g-k)$ of the partition.

Theorem 1.3. *Fix any $k \geq 1$. Then*

$$\lim_{g\to\infty} \frac{N(g, 2g-k)}{N(g)} = 0.$$

Finally, in Section 5, we study the local behavior of $S(g)$ and $N(g, 2g-k)$, and state conjectures about the growth of each value depending on the residue of $g \pmod 3$.

2 Preliminaries

We begin with a few definitions.

Definition 2.1. Given a numerical semigroup Γ, the set of elements of Γ that cannot be expressed as a sum of nonnegative multiples of smaller elements is called the *minimal set of generators* of Γ. Note that this set is finite and unique.

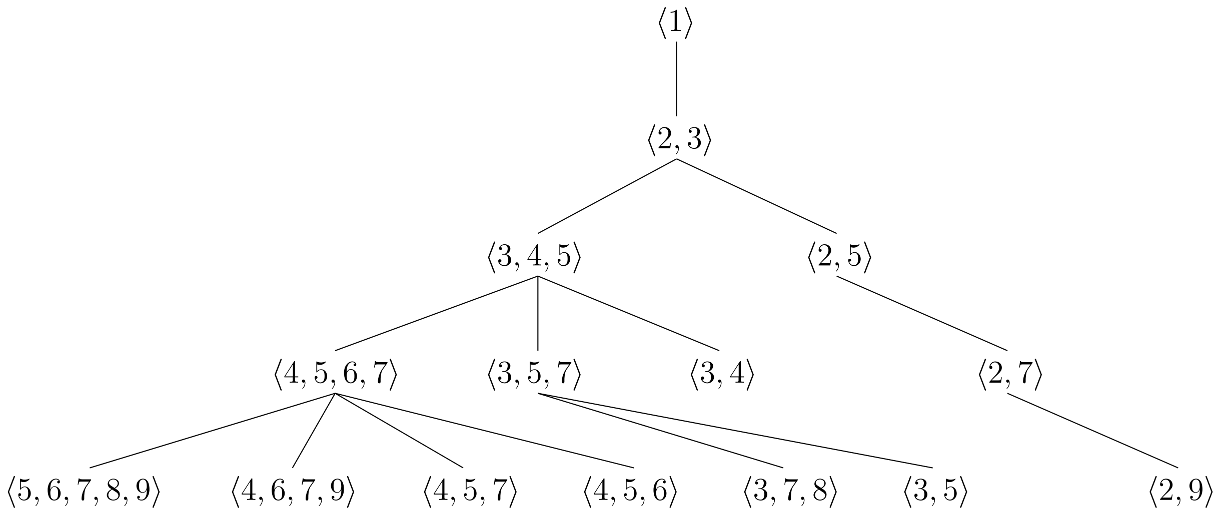

Figure 1: Tree containing all numerical semigroups.

Definition 2.2. Given a numerical semigroup Γ, the smallest minimal generator is called the *multiplicity* of Γ, and is denoted $m = m(\Gamma)$.

Example 2.3. The numerical semigroup $\{0, 3, 4\} \cup \mathbb{Z}_{\geq 6}$ has gaps $1, 2$, and 5, so the genus is 3 and the Frobenius number is 5. The minimal set of generators is $\{3, 4\}$, so the multiplicity is 3.

We denote by $\langle a_1, a_2, \ldots, a_n \rangle$ the numerical semigroup generated by $\{a_1, a_2, \ldots, a_n\}$. Then, we recursively define a partial ordering on the set of numerical semigroups as follows. (We refer the reader to Sections 3.1–3.3 of [7] for background on partially ordered sets and trees.)

Let rank 0 contain only $\mathbb{N}_0 = \langle 1 \rangle$. Then for any two semigroups Γ_1 and Γ_2, let Γ_1 cover Γ_2 if Γ_2 can be obtained by removing exactly one element of Γ_1 greater than $F(\Gamma_1)$. Thus, rank g contains precisely the semigroups of genus g. The first five ranks of the tree are depicted in Figure 1.

We define a *leaf* to be any semigroup with no children in the semigroup tree. The leftmost semigroup on each rank of the tree in Figure 1 is *ordinary*, or of the form $\{0\} \cup \{i \in N_0 \mid i \geq c\}$. The following key lemma about the covering relations of the semigroup tree is a direct consequence of the proof of Lemma 3.1 in [3].

Lemma 2.4. *Let Γ be a non-ordinary numerical semigroup with minimal generators $\Lambda = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k\}$, and suppose that it has Frobenius number F such that $\lambda_i \leq F \leq \lambda_{i+1}$. Then its children in the semigroup tree are precisely $\{\Gamma \setminus \{\lambda_j\} \mid j > i\}$. Furthermore, each semigroup $\Gamma \setminus \{\lambda_j\}$ is generated by $\Lambda \setminus \{\lambda_j\}$ or $\{\lambda_1 + \lambda_j\} \cup \Lambda \setminus \{\lambda_j\}$.*

Proof. We closely follow the proof from [3]. Any minimal generator of $\Gamma \setminus \{\lambda_j\}$ is either a minimal generator of Γ or of the form $\lambda_j + r$ for some $r \in \Gamma$. If $r > \lambda_1$, then $s = \lambda_j + r - \lambda_1 \in \Gamma$ because λ_j is greater than the Frobenius number. Thus, $\lambda_j + r$ is not a minimal generator of $\Gamma \setminus \{\lambda_j\}$, so the only element that may be a generator of $\Gamma \setminus \{\lambda_j\}$ but not Γ is $\lambda_j + \lambda_1$. Finally, notice that any minimal generator λ_i of Γ other than λ_j must also be a generator of $\Gamma \setminus \{\lambda_j\}$ since λ_i, by definition, cannot be expressed as a sum of multiples of smaller elements of $\Gamma \supset \Gamma \setminus \{\lambda_j\}$. $\square$

Notice that Lemma 2.4 only applies to non-ordinary numerical semigroups. Thus, we consider the special case of ordinary semigroups separately as follows.

Lemma 2.5. *Let $\Gamma = \langle g+1, g+2, \ldots, 2g, 2g+1 \rangle$ be the unique ordinary numerical semigroup with g gaps, and let Λ be the set of its minimal generators. Then its children in the semigroup tree are precisely $\{\Gamma \setminus \{g+k\} \mid 1 \leq k < g+1\}$. Furthermore, each semigroup $\Gamma \setminus \{g+k\}$ where $2 \leq k < g+1$ is generated by $\Lambda \setminus \{g+k\}$ or $\{2g+k\} \cup \Lambda \setminus \{g+k\}$, and the semigroup $\Gamma \setminus \{g+1\}$ is generated by $\{g+2, g+3, \ldots, 2g+2, 2g+3\}$.*

Proof. The proof is analogous to Lemma 2.4 with the exception of $\Gamma \setminus \{g+1\} = \{0\} \cup \mathbb{Z}_{\geq g+2}$. In this case, the minimal generators are clearly $\{g+2, g+3, \ldots, 2g+2, 2g+3\}$, so we are done. $\square$

Remark 2.6. The Frobenius number of $\Gamma \setminus \{\lambda_j\}$ for a numerical semigroup Γ is λ_j.

Lemma 2.4 shows that one way to construct the tree is to recursively find the children of any given semigroup. It also inspires the following definitions.

Definition 2.7. For any numerical semigroup Γ with minimal generators $\Lambda = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k\}$ and Frobenius number F, we call $\lambda_i \in \Lambda$ an *effective generator* if $\lambda_i > F$.

Definition 2.8. The *efficacy* of a numerical semigroup Γ, denoted $h(\Gamma)$, is the number of effective generators of Γ.

Definition 2.9. Suppose the semigroup Γ has parent Γ'. If the minimal generators of Γ are strictly contained in the set of minimal generators of Γ', then we say that Γ is *weakly descended*. Otherwise, we call Γ *strongly descended*.

We provide an example of the procedure described in Lemma 2.4 below.

Example 2.10. Consider the semigroup $\Gamma = \langle 3, 5, 7 \rangle$. The Frobenius number of Γ is 4, so it has effective generators 5 and 7, and multiplicity $\lambda_1 = 3$. Now $\Gamma \setminus \{5\}$ is generated by $\{3, 7, 3+5\} = \{3, 7, 8\}$, and $\Gamma \setminus \{7\}$ is generated by $\{3, 5\}$. Notice that, since $3 + 7 = 10$ already belongs to the semigroup $\langle 3, 5 \rangle$, it is not a minimal generator of $\Gamma \setminus \{7\}$. Thus, the children of $\langle 3, 5, 7 \rangle$ are $\langle 3, 7, 8 \rangle$ and $\langle 3, 5 \rangle$.

Notice that every numerical semigroup with g gaps appears on rank g of the semigroup tree. We demonstrate this briefly using induction. Clearly, every semigroup with 0 gaps (only $\langle 1 \rangle$) appears on rank 0. Now assume that any numerical semigroup with $g-1$ gaps belongs to rank $g-1$ of the tree. Consider any semigroup Γ with g gaps and Frobenius number F. Then the semigroup $\Gamma \cup F$ has $g-1$ gaps. Furthermore, F must be an effective generator of Γ because, by definition, F cannot be obtained by summing elements of Γ, and every integer greater than F already belongs to Γ. Thus, Γ is a child of $\Gamma \cup F$, and by induction, belongs to the semigroup tree on rank g. The number of semigroups on each rank of the tree has been studied by a number of mathematicians, as discussed in Section 1.

Now, we describe a family of numerical semigroups known as *symmetric semigroups*, which are a central focus of this paper.

Definition 2.11. Suppose the semigroup Γ has Frobenius number F. Then Γ is *symmetric* if it contains exactly $\frac{F+1}{2}$ elements less than or equal to F.

We provide an example of a symmetric semigroup below.

Example 2.12. The semigroup $\langle 3, 4 \rangle$ has Frobenius number 5, and contains $0, 3,$ and 4, but not $1, 2,$ or 5. Thus, it contains precisely half of the elements less than or equal to its Frobenius number, so it is symmetric.

Symmetric semigroups arise naturally from the definition of a numerical semigroup. If a semigroup has Frobenius number F, then the semigroup may only contain one element of each pair $(k, F-k)$ for every $0 \leq k \leq F$. Thus, a semigroup may contain at most half of the positive integers below or equal to its Frobenius number, and symmetric semigroups are defined to contain exactly half. Furthermore, the number symmetric semigroups in $N(g)$ minus one provides a lower bound for the number of leaves in $N(g)$ [1].

Lemma 2.13 ([1], Lemma 4)**.** *Every symmetric numerical semigroup is either a leaf or of the form* $\langle 2, 2n+1 \rangle$ *for some* $n \in \mathbb{Z}_{\geq 1}$.

3 Bounding Symmetric Semigroups

In this section, we explore the number of symmetric semigroups with g gaps, denoted $S(g)$. We begin by comparing the cardinality of symmetric semigroups of genus g and semigroups with genus $g-1$ and minimal generator $2g-1$.

Lemma 3.1. *The number of semigroups with* $g-1$ *gaps such that* $2g-1$ *is a minimal generator is precisely* $S(g)$.

Proof. Given any semigroup Γ and any effective generator λ_j of Γ, the Frobenius number of $\Gamma \setminus \{\lambda_j\}$ is λ_j. Since any symmetric semigroup of genus g has Frobenius number $2g-1$, it can be expressed as $\Gamma \setminus \{2g-1\}$, where Γ has genus $g-1$ and $2g-1$ is an effective generator. Conversely, notice that the maximum possible Frobenius number of a numerical semigroup on rank $g-1$ is $2g-3$ (else two elements would necessarily sum to the Frobenius number). Thus, if $2g-1$ is a minimal generator of Γ, then it is an effective generator, so $\Gamma \setminus \{2g-1\}$ is a symmetric semigroup. This completes the proof. $\square$

We can also relate $S(g)$ to the number of semigroups on rank $g-2$, denoted $N(g-2)$.

Lemma 3.2. *For all $g \geq 2$, the inequality $S(g) \leq N(g-2)+1$ holds.*

Proof. We claim that every non-ordinary numerical semigroup has at most one symmetric grandchild. By Lemma 3.1, any symmetric semigroup with g gaps is obtained from its parent by removing $2g-1$. Thus, no semigroup can have two symmetric children. Now suppose that a semigroup Γ has a symmetric grandchild with g gaps. Then Γ must have a child Γ_1 with minimal generator $2g-1$. However, $2g-1$ cannot be a minimal generator of Γ (else Γ would have a child with $g-1$ gaps and Frobenius number $2g-1$). Thus, Γ must contain the minimal generator $2g-1-\lambda_1 > F$, where λ_1 is the smallest element of Γ, and F is the Frobenius number of Γ. Thus, a maximum of one of the children of Γ contains $2g-1$ as a minimal generator, so Γ has at most one symmetric grandchild.

Now consider an ordinary semigroup Γ. In this case, we claim that Γ has exactly two symmetric grandchildren, with the exception of $\Gamma = \langle 1 \rangle$, in which case there is only one. If $\Gamma \neq \langle 1 \rangle$, then by Lemma 2.5, precisely two of its children contain $2g-1$ as a minimal generator. Thus, exactly two of its grandchildren are symmetric. If $\Gamma = \langle 1 \rangle$, it is clear that it has exactly one symmetric grandchild.

Notice that there is exactly one ordinary semigroup on each rank. The inequality $S(g) \leq N(g-2)+1$ easily follows. $\square$

Finally, we prove that $S(g)$ vanishes with respect to $N(g)$.

Theorem 3.3. *We have*

$$\lim_{g\to\infty} \frac{S(g)}{N(g)} = 0.$$

Before we state the proof, we must summarize Zhai's argument in [8]. He proves the following theorem.

Theorem 3.4 (Zhai)**.** *Let ϕ represent the golden ratio. Then*

$$\lim_{g\to\infty} \frac{N(g)}{\phi^g} = C$$

for some constant C.

Let T denote the set of strongly descended numerical semigroups. Let $N_g(\Gamma)$ denote the number of weakly descended semigroups of Γ with genus g. In other words, $N_g(\Gamma)$ is the number of semigroups with genus g whose nearest strongly descended ancestor is Γ. Since every numerical semigroup is weakly descended from exactly one strongly descended semigroup, we have

$$N(g) = \sum_{\Gamma \in S} N_g(\Gamma).$$

Zhai partitions T into the following three subsets:

$$\begin{aligned}
T_1 &= \{\Gamma \in S \mid h(\Gamma) + g(\Gamma) < g\} \\
T_2 &= \left\{\Gamma \in S \mid h(\Gamma) + g(\Gamma) \geq g \text{ and } g(\Gamma) - h(\Gamma) < \frac{g}{3}\right\} \\
T_3 &= \left\{\Gamma \in S \mid h(\Gamma) + g(\Gamma) \geq g \text{ and } g(\Gamma) - h(\Gamma) \geq \frac{g}{3}\right\}.
\end{aligned}$$

Manipulating the inequalities defining T_2 yields

$$3(g(\Gamma) - h(\Gamma)) < g \leq g(\Gamma) + h(\Gamma) \tag{1}$$

$$g(\Gamma) \leq 2h(\Gamma). \tag{2}$$

Zhai introduces the following definition using these inequalities.

Definition 3.5. A numerical semigroup Γ satisfying equations (1) and (2) is called *orderly*.

He then proves the following (in Proposition 1 of [8]).

Lemma 3.6. *If Γ is orderly, then $F(\Gamma) < 2m(\Gamma)$.*

Denote by $N(g,i)$ the number of numerical semigroups in $N(g)$ and T_i. Zhai proves (in Section 3 of [8]) that $N(g,1) = 0$ and that

$$\lim_{g \to \infty} \frac{N(g,3)}{N(g)} = 0. \tag{3}$$

We demonstrate the following.

Lemma 3.7. *Given any $g \geq 2$, there is precisely one semigroup of genus g that is both symmetric and orderly.*

Proof. Suppose a symmetric semigroup Γ has multiplicity m and genus $g \geq 2$. Then it has at least m gaps. Since Γ is symmetric, the minimum possible Frobenius number is $2m-1$. By Lemma 3.6, any orderly semigroup satisfies $F(\Gamma) < 2m(\Gamma)$. Thus, we must have $F(\Gamma) = 2m-1$. To satisfy the symmetric condition, Γ must contain every integer between $m(\Gamma)$ and $F(\Gamma)$. Thus, Γ is must be of the form $\{0, m+1, m+2, \dots, 2m-2, 2m, 2m+1, 2m+2, \dots\}$. Notice that Γ has precisely $m+1$ gaps. Thus, given $g \geq 2$, there exists exactly one symmetric and orderly semigroup with g gaps, as desired. $\square$

Recall that every orderly semigroup belongs to T_2, and that T_1 is empty. Then Lemma 3.7 implies that every symmetric semigroup, except for one on each rank, belongs to T_3. Thus, equation (3) implies

$$0 \leq \lim_{g\to\infty} \frac{S(g)}{N(g)} \leq \lim_{g\to\infty} \frac{N(g,3)+1}{N(g)} = 0.$$

This completes the proof of Theorem 3.3.

4 Bounding Generalized Symmetric Semigroups

Recall that any symmetric semigroup with genus g has Frobenius number $2g-1$. This motivates the following generalization.

Let $N(g, 2g-k)$ denote the number of semigroups with genus g and Frobenius number $2g-k$, where $1 \leq k \leq g$. Notice that $k = 1$ represents the number of symmetric semigroups. Likewise, $k = g$ represents the number of ordinary semigroups with g gaps, which is precisely 1. Notice that $N(g, 2g-k)$, for $1 \leq k \leq g$, forms a partition of $N(g)$.

Lemma 4.1. *Given any $g \geq 1$,*

$$\sum_{k=1}^{g} N(g, 2g-k) = N(g).$$

Proof. A semigroup with g gaps clearly has Frobenius number $F \geq g$. Furthermore, if the Frobenius number $F \geq 2g$, then the semigroup must contain at least one of the pairs $\{n, F-n\}$ for some $0 \leq n \leq F$, a contradiction. Thus, $g \leq F \leq 2g-1$, as desired. $\square$

We can extend 3.3 to $N(g, 2g-k)$ for any fixed value of k.

Theorem 4.2. *Fix $k \geq 1$. Then*

$$\lim_{g\to\infty} \frac{N(g, 2g-k)}{N(g)} = 0.$$

Proof. Fix $k \geq 1$. Similarly to the proof of Theorem 3.3, we aim to count the number of orderly semigroups with g gaps and Frobenius number $F = 2g - k$. Consider any such semigroup Γ, and suppose that it has multiplicity m. By Lemma 3.6, we must have $2m > F$. Then $m \leq g + 1$, so $F = 2g - k \geq 2m - k - 2$. So far, we have bounded F above and below:

$$2m - k - 2 \leq F \leq 2m - 1.$$

For simplicity, assume that Γ is not ordinary. (We can ignore this case because there is exactly one ordinary semigroup with g gaps for every $g \geq 0$.) Then the inequality becomes $2m - k \leq F \leq 2m$. Notice that $F = 2g - k$ and k have the same parity. Thus, we can write $F = 2m - k + 2n$, for $2n$ in the range $0 \leq 2n \leq k - 1$. We have $F = 2m - k + 2n = 2g - k$, which implies that $m + n = g$. Now note that Γ contains 0 and m, but excludes the integers $1, 2, \ldots, m - 1$ and the Frobenius number $2m - k + 2n$. Thus, there must be exactly $g - m = n$ gaps of Γ within the range $(m, 2m - k + 2n)$, which yields a maximum of $\binom{m-k+2n-1}{n} = \binom{g-k+n-1}{n}$ possibilities. We can sum over all possible values of n to find an upper bound for the number of semigroups with g gaps and Frobenius number $2g - k$:

$$N(g, 2g - k) \leq \sum_{0 \leq n \leq \frac{k-1}{2}} \binom{g - k + n - 1}{n} + 1.$$

(Recall that we ignored ordinary semigroups; to account for this, we add one to the right hand side of the inequality.) Since n varies within a fixed range independent of g, this expression is polynomial in g. Because $N(g)$ grows exponentially, we must have

$$\lim_{g \to \infty} \frac{N(g, 2g - k)}{N(g)} = 0,$$

as desired.

$\square$

5 Local Growth of $S(g)$ and $N(g, 2g - k)$

So far, we have studied the global growth of $S(g)$ and $N(g, 2g - k)$ as g goes to infinity. Now, we turn to the patterns displayed by both quantities on a local scale.

First, consider the sequence $S(0), S(1), S(2), \ldots$. Define the sequence $a_0, a_1, a_2, \ldots$ such that

$$a_g = \frac{S(g + 1)}{S(g)}.$$

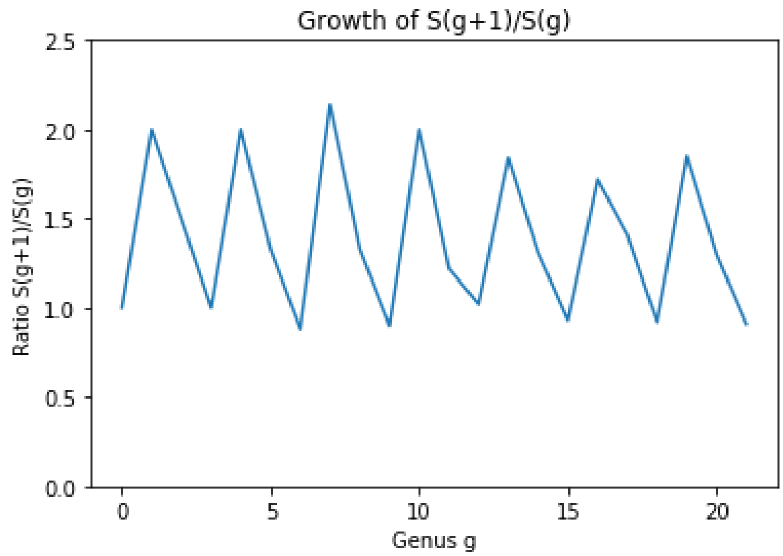

Figure 2: Growth of $\frac{S(g+1)}{S(g)}$

The graph in Figure 2 demonstrates a pattern in the values of a_g that correlates with the residue of $g \pmod 3$. Thus, it is natural to ask about the ratio $\frac{S(g+3)}{S(g)}$. This ratio, graphed in Figure 3, appears to converge. This suggests that the growth of $S(g)$ is exponential. We state these observations more formally as follows.

Conjecture 5.1. *For an appropriate constant C, $\lim_{g\to\infty} \frac{S(g+3)}{S(g)} = C$.*

Conjecture 5.2. *For any $g \geq 0$, the inequality $a_{3g} < a_{3g+2} < a_{3g+1}$ holds.*

We can extend this conjecture from $S(g)$ to $N(g, 2g-k)$. Graphed in Figure 4 is the ratio

$$b_g = \frac{N(g+1, 2(g+1)-k)}{N(g, 2g-k)},$$

where every color represents a fixed value of k. (For clarity, we have also included the same graph with only $k = 1, 2, 3$ in Figure 5.)

Notice that the sequence b_g for each value of k appears to be bounded above by its first peak. We can precisely state the value of each of these peaks as follows.

Lemma 5.3. *Fix $k \geq 1$. Let g_k be the first rank that contains an element with Frobenius number $2g-k$. Then*

$$\frac{N(g_k+1, 2(g_k+1)-k)}{N(g_k, 2g_k-k)} = \left\lceil \frac{k}{2} \right\rceil.$$

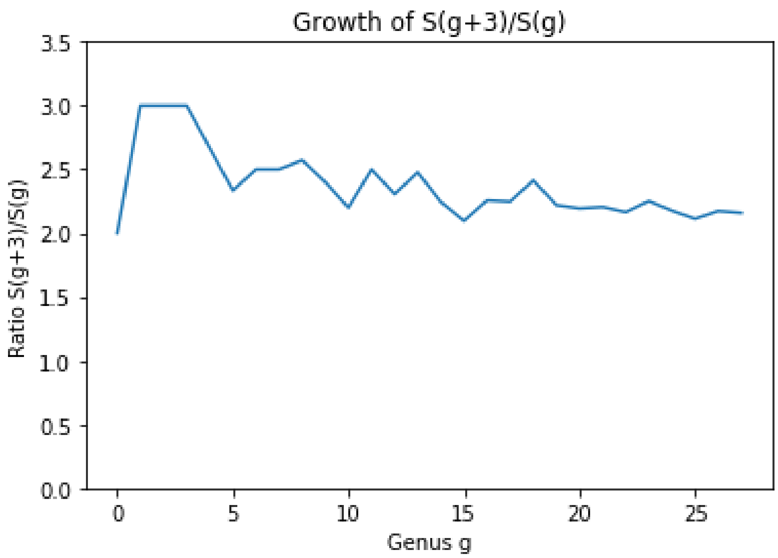

Figure 3: Growth of $\frac{S(g+3)}{S(g)}$

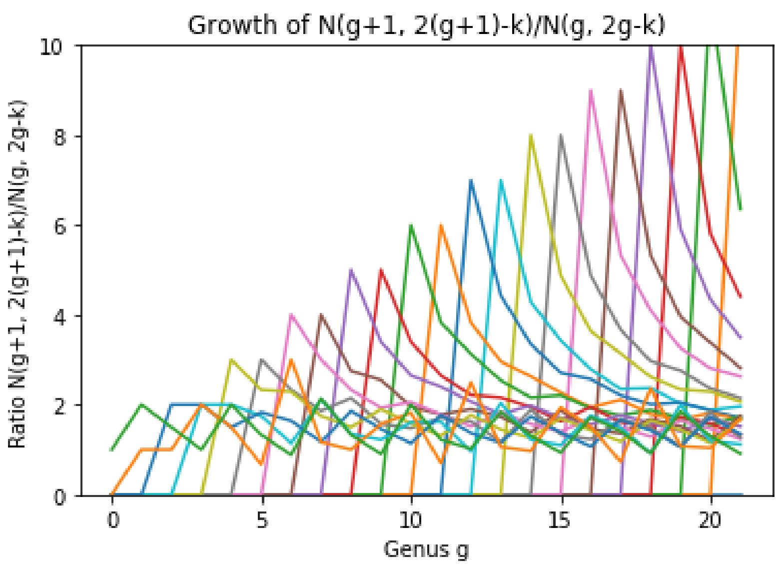

Figure 4: Growth of $\frac{N(g+1,2(g+1)-k)}{N(g,k)}$

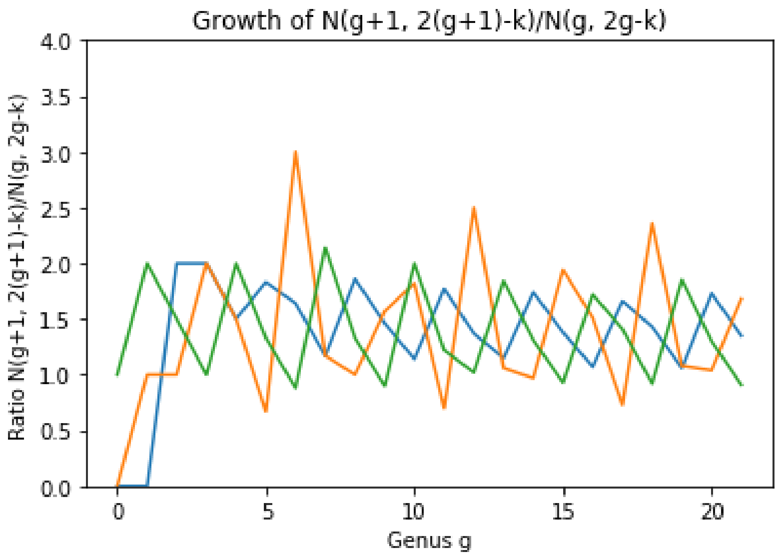

Figure 5: Growth of $\frac{N(g+1,2(g+1)-k)}{N(g,k)}$ for $k = 1, 2, 3$

Proof. Notice that $g_k = k$, since the ordinary semigroup has the minimum possible Frobenius number when the genus is fixed. Then every semigroup with $g_k+1 = k+1$ gaps and Frobenius number $2(g_k + 1) - k = k + 2$ must contain exactly one integer between 0 and $k + 2$; in other words, it must be of the form $\{0, m, k + 3, k + 4, k + 5, \ldots\}$, where $0 < m < k + 2$. However, to preserve closure under addition, we must have $m > \frac{k+2}{2}$. Thus, there are a total of $k + 1 - \left\lfloor \frac{k+2}{2} \right\rfloor = \left\lceil \frac{k+2}{2} \right\rceil - 1 = \left\lceil \frac{k}{2} \right\rceil$ possibilities for m. Thus, $\frac{N(k+1,k+2)}{N(k,k)} = \left\lceil \frac{k}{2} \right\rceil$, as desired. $\square$

Finally, notice that each sequence corresponding to a fixed k appears to converge to a pattern, again depending on the residue of $g \pmod 3$. We state this conjecture as follows.

Conjecture 5.4. *Given any $k \geq 1$, there exists $n \geq 1$ such that, for all $g \geq 0$, the inequality $b_{3g+n} \leq b_{3g+n+2} \leq b_{3g+n+1}$ holds.*

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