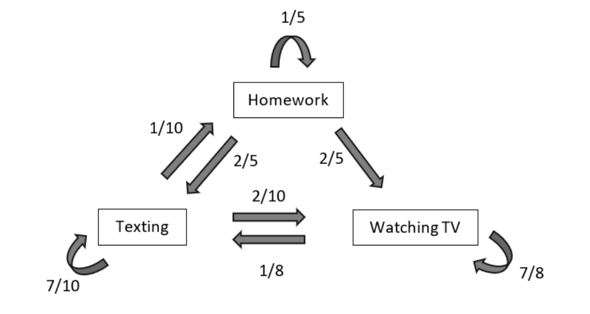
Jumping Into Markov Chains

A PRIMES Exposition

Ishita Goluguri, Toyesh Jayaswal
Mentor: Dr. Benjamin Landon

TABLE OF CONTENTS

- A Markov process is characterized by the memoryless property that the future only depends on the current state and not on the previous states.
- For example, a student may follow the chain below every 15 minutes.



- Another example is radioactive decay where the time before the next particle decays does not depend on when the previous particles decayed.

Definition (State Space)

The state space I is the set of all possible states of the Markov Chain.

Definition (Measure and Distribution)

A measure is a row vector $\lambda = (\lambda_i : i \in I)$ taking non-negative values in $\mathbb{R}$. A distribution is a measure with $\sum \lambda_i = 1$.

Definition (Transition Matrix P)

$P = (p_{ij} : i, j \in I)$, where p_{ij} is the probability of jumping from state i to state j. $p_{ij}^{(n)}$ is the probability of transitioning from i to j in n steps and is the ij entry of P^n.

Definition (Markov Chain)

A sequence of random variables X_n taking values in I is Markov(λ, P) if $\mathbb{P}(X_0 = i) = \lambda_i$ and $\mathbb{P}(X_{n+1} = j | X_n = i) = p_{ij}$.

Definition

For a certain subset $A \subset I$ and state $i \in I$, the hitting probability is defined as $h_i^A = \mathbb{P}_i(\text{hit A})$ and the hitting time is defined as $k_i^A = \mathbb{E}_i(\text{time to hit A})$.

The hitting probabilities satisfy

$$\begin{cases} h_i^A = 1 & i \in A \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A & i \notin A \end{cases}$$

The hitting times satisfy

$$\begin{cases} k_i^A = 0 & i \in A \\ k_i^A = 1 + \sum_{j \in I} p_{ij} k_j^A & i \notin A \end{cases}$$

Moreover, h_i^A and k_i^A are the minimal non-negative solutions to these equations.

Proof for Hitting Probabilities

$$\begin{cases} h_i^A = 1 & i \in A \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A & i \notin A \end{cases}$$

If $i \in A$, $h_i^A = 1$ trivially

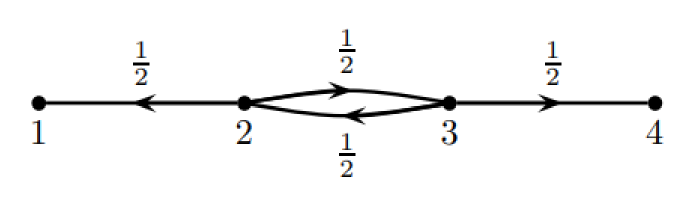
for $i \notin A$, let $H^A(\omega) = \inf\{n | X_n(\omega) \in A\}$.

$$h_i^A = \mathbb{P}_i(H^A < \infty)$$

$$\mathbb{P}_i(H^A < \infty | X_1 = j) = \mathbb{P}_j(H^A < \infty)$$

$$h_i^A = \sum_{j \in I} p_{ij} \mathbb{P}_j(H^A < \infty) = \sum_{j \in I} p_{ij} h_j^A$$

A Simple Example



In the chain above, what is the probability of getting to state 4 starting from state 2?

$$h_4 = 1$$

$$h_1 = h_1$$

$$h_2 = h_1/2 + h_3/2$$

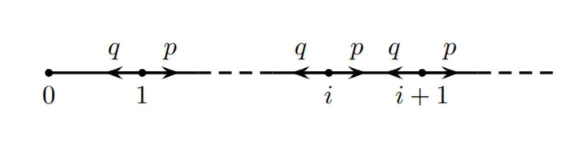
$$h_3 = h_2/2 + h_4/2$$

$$h_2 = 1/3$$

Gamblers’ Ruin

Problem

Imagine that you enter a casino with a fortune of \$$i$ and gamble, \$1 at a time, with probability p of doubling your stake and probability q of losing it. What is the probability that you leave broke?



Let $h_i = \mathbb{P}_i$(hitting 0).
We get the system $h_0 = 1, h_i = ph_{i+1} + qh_{i-1}$.
General solution: $h_i = A + B\left(\frac{q}{p}\right)^i$ If $p < q$, then $B = 0$, so $h_i = 1$. Similarly, if $p = q$, then $h_i = A + Bi$, and $B = 0$ once again. Thus, $h_i = 1$.
If $p > q$, then $h_i = \left(\frac{q}{p}\right)^i + A\left(1 - \left(\frac{q}{p}\right)^i\right)$, with the minimal nonnegative solution being $h_i = \left(\frac{q}{p}\right)^i$

Definition (Recurrence)

A state i is recurrent if $\mathbb{P}_i(\{t \geq 0 : X_t = i\}$ is unbounded) = 1

Definition (Transience)

A state i is transient if $\mathbb{P}_i(\{t \geq 0 : X_t = i\}$ is unbounded) = 0

Definition (Communicating States)

State i communicates with state j if $\mathbb{P}_i(X_n = j$ for $n \geq 0) > 0$ and $\mathbb{P}_j(X_n = i$ for $n \geq 0) > 0$

Communicating is an equivalence relation, and partitions the state space into communicating classes. If I is a single class, P is said to be irreducible.

Definition (Closed Communicating Class)

A communicating class is closed if $i \in C$ and i communicates with j implies that $j \in C$

Properties of Recurrence and Transience

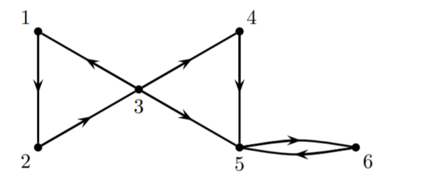


Figure: Communicating classes : {1,2,3}, {4}, {5, 6}

Definition (First Passage Time to State i)

$$T_i(\omega) = \inf\{n \geq 1 : X_n(\omega) = i\}$$

Definition (Return probability)

$$f_i = \mathbb{P}_i(T_i < \infty)$$

Definition (Number of visits V_i)

$$V_i = \sum_{n=0}^{\infty} 1_{X_n = i}, \mathbb{E}_i(V_i) = \sum_{n=0}^{\infty} p_{ii}^{(n)}$$

Theorem

- *If $f_i = 1$, then i is recurrent and $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$*
- *If $f_i < 1$, then i is transient and $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$*

Proof.

If $\mathbb{P}_i(T_i < \infty) = 1$, then $\mathbb{P}_i(V_i = \infty) = 1$, so i is recurrent, and $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \mathbb{E}_i(V_i) = \infty$.
If $f_i = \mathbb{P}_i(T_i < \infty) < 1$, then

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = \mathbb{E}_i(V_i) = \sum_{r=0}^{\infty} \mathbb{P}_i(V_i > r) = \sum_{r=0}^{\infty} f_i^r = \frac{1}{1 - f_i} < \infty$$

□

Theorem

All states in a communicating class are either recurrent or transient

Proof.

Take $i, j \in C$ and assume i is transient.Thus, there exist $n, m \geq 0$ such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$. For all $r \geq 0$,

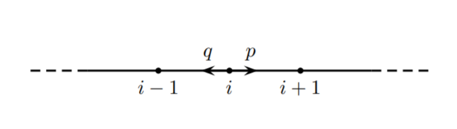
$$p_{ii}^{(m+n+r)} \geq p_{ij}^{(n)} p_{jj}^{(r)} p_{ji}^{(m)},$$

so

$$\sum_{r=0}^{\infty} p_{jj}^{(r)} \leq \frac{1}{p_{ij}^{(n)} p_{ji}^{(m)}} \sum_{r=0}^{\infty} p_{ii}^{(n+r+m)} < \infty$$

□

Random Walks on $\mathbb{Z}$



Given an odd sequence, $p_{00}^{(2n+1)} = 0$ for all n.
Given an even sequence of length $2n$, the probability of having n steps up and n steps down is $\binom{2n}{n} p^n q^n$.
By Stirling's formula,

$$n! \approx \sqrt{2\pi n}(n/e)^n \text{ as } n \to \infty$$

Thus,

$$p_{00}^{(2n)} = \frac{(2n)!}{(n!)^2}(pq)^n \approx \frac{(4pq)^n}{A\sqrt{n/2}} \text{ as } n \to \infty$$

For $p = q = \frac{1}{2}$,

$$p_{00}^{(2n)} \geq \frac{1}{2\sqrt{2\pi n}}, \text{ so } \sum_{n=N}^{\infty} p_{00}^{(2n)} \geq \frac{1}{2\sqrt{2\pi}} \sum \frac{1}{\sqrt{n}} = \infty$$

so the random walk on $\mathbb{Z}$ is recurrent.

If $p \neq q, 4pq < 1$, so

$$\sum_{n=N}^{\infty} p_{00}^{(2n)} \leq \frac{1}{\sqrt{2\pi}} \sum_{n=N}^{\infty} (4pq)^n < \infty$$

so this walk is transient.

$$p_{ij} = \begin{cases} \frac{1}{6} & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Again, with an odd step sequence, $p_{00}^{(2n+1)} = 0.$
With an even step sequence, we must have i steps up and down, j steps north and south, k steps east and west with $i + j + k = n.$

$$\begin{aligned} p_{00}^{(2n)} &= \sum_{i+j+k=n} \frac{(2n)!}{(i!j!k!)^2} \left(\frac{1}{6}\right)^{2n} \\ &= \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \sum_{i+j+k=n} \binom{n}{ijk} \left(\frac{1}{3}\right)^{2n} \end{aligned}$$

For $n = 3m$, $\binom{n}{ijk} \leq \binom{n}{mmm}$, so

$$p_{00}^{(2n)} \leq \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \binom{n}{mmm} \left(\frac{1}{3}\right)^{n} \approx \frac{1}{2\sqrt{2\pi}^3} \left(\frac{6}{n}\right)^{3/2} \text{ as } n \to \infty$$

$\sum_{n=0}^{\infty} p_{00}^{(6m)} < \infty$ because $\sum n^{-3/2}$ converges. Since $p_{00}^{(6m)} \geq \left(\frac{1}{6}\right)^2 p_{00}^{(6m-2)}$ and $p_{00}^{(6m)} \geq \left(\frac{1}{6}\right)^4 p_{00}^{(6m-4)}$,

$$\sum_{n=0}^{\infty} p_{00}^{(n)} < \infty$$

and this walk is transient.

A measure π is called invariant if $\pi P = \pi$.

For a process X which is Markov(π, P) for an invariant distribution π, X_n is also Markov(π, P) for all n.

For a fixed state k, let $\gamma_i^k = \mathbb{E}_k \sum_{n=0}^{T_k - 1} 1_{\{X_n = i\}}$ be the expected number of visits to i between visits to k. γ^k turns out to be an invariant measure with $\gamma_k^k = 1$.

If $\sum_i \gamma_i^k = m_k$, which is the expected return time to k, is finite (positive recurrence), γ^k / m_k is an invariant distribution.

If a chain is irreducible and positive recurrent, the invariant measure turns out to be unique up to scaling and in this case, $\pi_k = \frac{1}{m_k}$.

Theorem (Convergence to Equilibrium)

If P is irreducible and aperiodic, $\mathbb{P}(X_n = j) \to \pi_j$ as $n \to \infty$ for all j regardless of the initial distribution.

Periodic case: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Ergodic Theorem

Theorem (Ergodic Theorem)

Let P be irreducible and positive recurrent and let X be Markov(λ, P). Then, for any bounded function $f : I \to \mathbb{R}$,

$$\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n-1} f(X_k) \to \bar{f} \text{ as } n \to \infty\right) = 1$$

where $\bar{f} = \sum_{i \in I} \pi_i f(i)$ regardless of the initial distribution.

Example of an Invariant Distribution

An opera singer is due to perform a long series of concerts. She is liable to pull out each night with probability $1/2$. The promoter sends her flowers every day until she returns. Flowers costing x thousand dollars, $0 \leq x \leq 1$, bring about a reconciliation with probability $\sqrt{x}$. The promoter stands to make $\$750$ from each successful concert. How much should he spend on flowers?

$$P = \begin{pmatrix} 1/2 & 1/2 \\ \sqrt{x} & 1 - \sqrt{x} \end{pmatrix}$$

$$\lambda_1 = \lambda_1/2 + \sqrt{x}\lambda_2$$

$$\lambda_2 = \lambda_1/2 + (1 - \sqrt{x})\lambda_2$$

$$\lambda_1 + \lambda_2 = 1$$

$$\lambda_1 = \frac{2\sqrt{x}}{2\sqrt{x}+1}, \lambda_2 = \frac{1}{2\sqrt{x}+1}$$

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \to 750\lambda_1 - 1000x\lambda_2 = \frac{1500\sqrt{x} - 1000x}{2\sqrt{x}+1}$$

$$x = 1/4, \mathbb{E}(f) \to \$250$$

Acknowledgements

- We would like to thank our parents for their endless support
- We would like to thank Dr Slava Gerovitch, Dr Tanya Khovanova, Dr Pavel Etingof, and the entire PRIMES program
- We would like to thank our mentor Dr. Benjamin Landon for teaching and mentoring us