A Brief Review of Special Topics in Group Theory

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Overview

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2. Symmetry Groups
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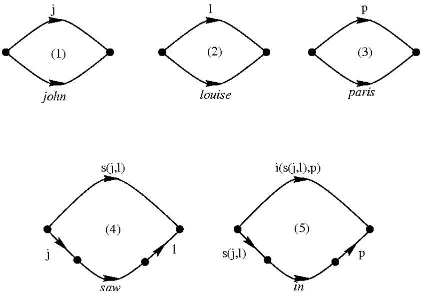
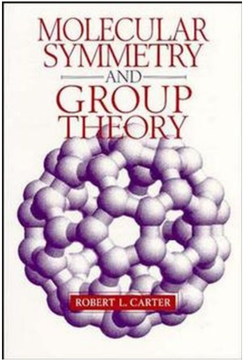
   - Definitions
   - C_n and D_n
   - Frieze and Wallpaper Groups
3. Tarski Groups
   - Burnside Problem
   - Cyclic Subgroups
   - Simple Group

Introduction

What is group theory and its applications?





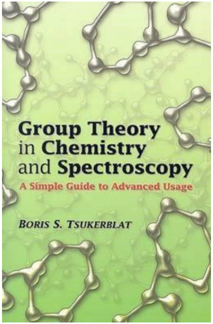


Image Source: Dymetman, M. (1998). Group Theory and Computational Linguistics. *Journal of Logic, Language and Information*, *7*, 461-497.

Groups

Definition 1.1

A *group* is an ordered pair $(G, \star)$, where G is a set and $\star$ is a binary operation on G that follows the following axioms:

i) The operation $\star$ is associative, for all $a, b, c \in G$, we have $(a \star b) \star c = a \star (b \star c)$

ii) The group G contains an element e, called the *identity*, such that for all $a \in G$, we have $a \star e = e \star a = a$

iii) For all $a \in G$, there is an element a^{-1}, called an *inverse*, such that $a \star a^{-1} = a^{-1} \star a = e$

Order

Definition 1.2

The *order* of a group, denoted by $|G|$, is the cardinality of the set G, the number of elements in it. G is an infinite group when its order is infinite.

Definition 1.3

The *order* of an element x in a group G, is the smallest $m \in \mathbb{Z}^+$ such that $x^m = e$ where e is the identity of G.

Subgroups

Definition 1.4

For a group G, we say that a subset H of G is a *subgroup*, denoted $H \leq G$, if H is a group under the operation of G. Trivial subgroup: $H = \{1\}$; Proper subgroups: all $H \neq G$.

Definition 1.5

A subgroup N is *normal*, denoted by $N \trianglelefteq G$, if the *conjugate* of N, $gNg^{-1} = \{gng^{-1} \mid n \in N\}$, is equal to N for all $g \in G$.

Alhambra (Granada, Spain)

Image Source: `mathstat.slu.edu`

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M.C. Escher

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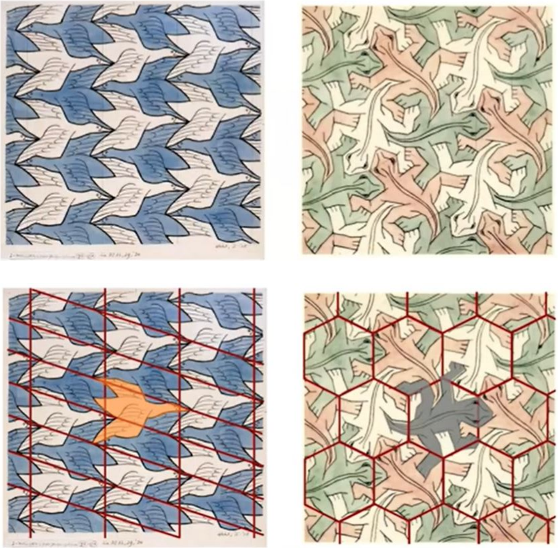

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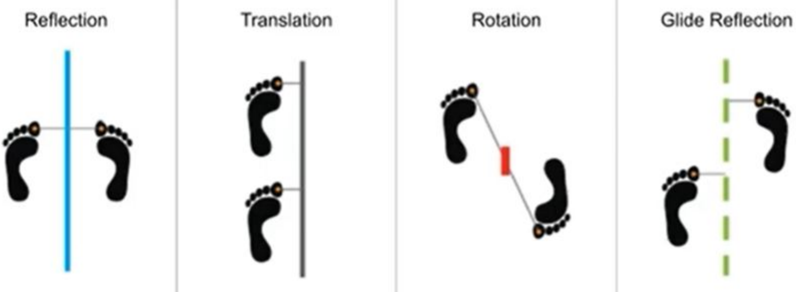
Transformations



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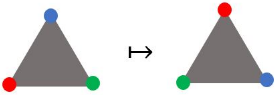
Symmetries

Definition 2.1

A *symmetry* is a transformation of the figure under which the figure is invariant.

Each symmetry g maps points of $F \mapsto F'$



120° rotation

Image Source: `youtu.be/uEBgG-Am0vA`

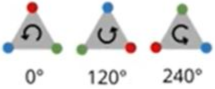
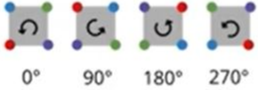
Symmetry Groups and Group Axioms

Definition 2.2

A *symmetry group* G is a set of all symmetries of a shape under the binary operation of composition of transformations.

- Binary operation is $\circ$
- *associativity*: $\circ$ is associative
- *identity*: e is symmetry that leaves all points unchanged
- *inverse*: inverse of a transformation is a composition of transformations

C_n

Symmetries	$n = 3$	$n = 4$
C_n Rotations		

$C_n = \{1, R^1, R^2, ..., R^{n-1}\}$ where R^i is a rotation of $\left(\frac{2\pi}{n}\right) i$ radians.

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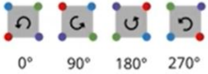
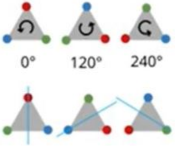
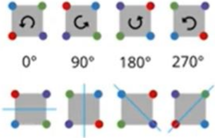
D_n

Symmetries	$n = 3$	$n = 4$
C_n Rotations		
D_n Rotations and Reflections		

- n rotations about the center, these are the elements of C_n
- n reflections through the n lines of symmetry.

Image Source: `youtu.be/uEBgG-Am0vA`

Frieze Patterns

Definition 2.3

A *frieze pattern* is a 2-dimensional design that repeats in 1 direction.

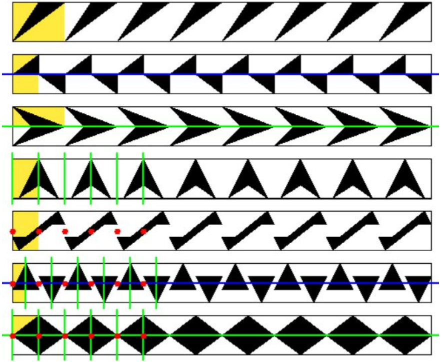


Image Source: `eecs.berkeley.edu/~sequin/`

Definition 2.4

Frieze groups are the symmetries of a frieze pattern.

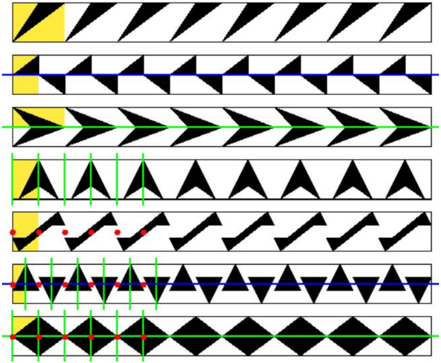


Image Source: `eecs.berkeley.edu/~sequin/`

Wallpaper Patterns

Definition 2.5

A *wallpaper pattern* is 2-dimensional design that repeats in 2 directions.

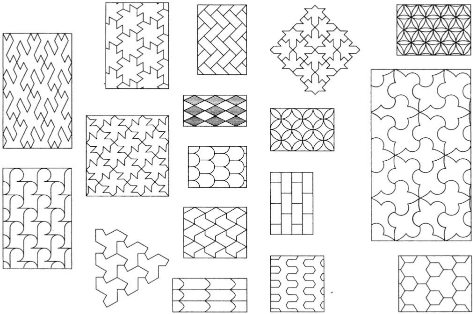


Image Source: Artin, Michael. *Algebra.* Pearson, 2011.

Definition 2.6

Wallpaper groups are the symmetries of a wallpaper pattern.

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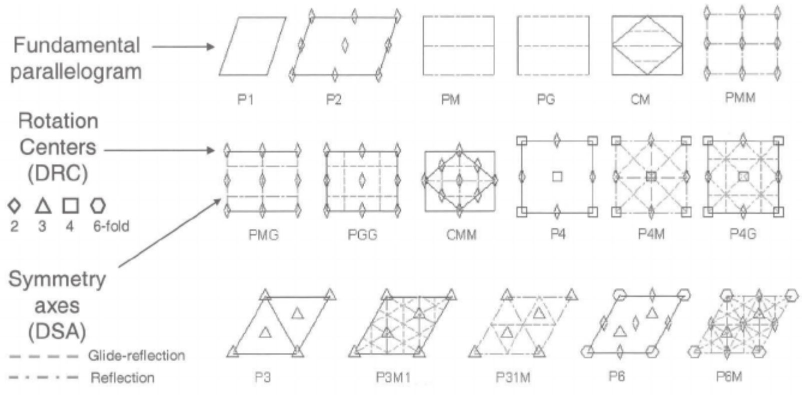

Image Source: Agustí-Melchor, Manuel & Rodas, Angel & González, José M. Computational Framework for Symmetry Classification of Repetitive Patterns. *Communications in Computer and Information Science*, 2013.

Tarski Groups

Definition 3.1 [Tarski Groups]

A *Tarski group* T is an infinite group where all nontrivial proper subgroups are of prime order.

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Existence

In a series of work published in the 1980s, Alexander Yu. Olshanskii has proven the existence of Tarski Monsters by constructing that the Tarski Monster exists for all primes $p > 10^{75}$.

Burnside Problem

Burnside Problem

Must a finitely generated group with elements of finite order be finite?

As Tarski Monsters are generated by two elements [Olshanskii], they suffice as counterexamples to the Burnside Problem.

Two Important Theorems

Theorem 3.3 [Cauchy]

For finite group G, there must be an element of order prime p if p divides $|G|$.

Theorem 3.4 [Lagrange]

If G is a finite group and H is a subgroup of G, the order of H divides the order of G.

Cyclic Groups and Generators

Definition 3.5

We say that a group G is *generated by* its subset S, denoted $G = \langle S \rangle$, if every element of G can be written as a finite product of elements in S. The elements of S are called *generators* of G.

Example 3.6

A cyclic group is generated by a single element: $H = \{x^n \mid n \in \mathbb{Z}\}$ for some $x \in H$.

Cyclic Subgroups

Theorem 3.7

All nontrivial proper subgroups of a Tarski Monster are cyclic.

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Proof.

Let H be a subgroup of a Tarski Monster. It must have cardinality prime p. By Cauchy's Theorem, there must be an element $x \in H$ that has order p. The p elements $x, x^2, \ldots, x^{p-1}, x^p = 1$ must be all distinct. For if $x^a = x^b$ for some $1 \leq a < b \leq p$, we get that $1 = x^{b-a}$ by the Cancellation Law, which contradicts that the order of x is p. And there are exactly p elements in H, so x is a generator of H and H is cyclic. □

Cyclic Subgroups

Corollary 3.8

For distinct proper subgroups H and K of T, we have $H \cap K = \{1\}$.

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For distinct proper subgroups H and K of T, we have $H \cap K = \{1\}$.

Proof.

For a cyclic group of prime order, any element that is not the identity can be the generator. This is true because the order of an element must divide the order of the group. In the case for prime p, all non-identity elements must have order p and thus can be a generator. Then, if there exists non-identity element $x \in H \cap K$, then $H = \langle x \rangle = K$. Hence, $H \cap K$ must only include 1. $\square$

The Subgroup Lattice

Definition 3.9

The *lattice of subgroups* of a group G is a plot of all subgroups as points positioned such that H is higher than K if $|H| > |K|$. In addition, a path is drawn from subgroups H to K if and only if $H \leq K$.

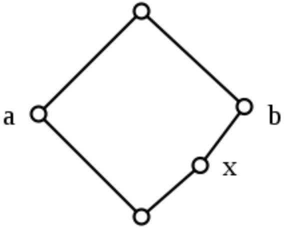
Modular lattices are a special type of lattice in that they are highly symmetric. They are rare among subgroup lattices of non-abelian groups.

The Subgroup Lattice

Theorem 3.10 [Dedekind]

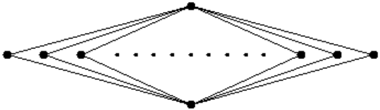
A lattice L is modular if and only if it does not contain N_5 as a sublattice.

The pentagon lattice N_5 is:



The Subgroup Lattice

The Tarski Monster lattice is modular:



In contrast, here is the subgroup lattice of C_{12}:

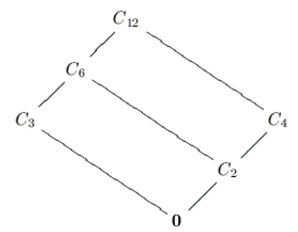


Image Source: `math.colostate.edu/~jwilson/math/12/C12.html`

Simple Groups

Definition 3.11

A group G is *simple* if $|G| > 1$ and its only normal subgroups are itself and the trivial subgroup.

Theorem 3.12

Tarski Monsters are simple groups.

Proving Tarski Monsters Are Simple

Lemma 3.13

If H and K are subgroups of group G, then $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G if and only if $HK = KH$.

Lemma 3.14

For finite subgroups H and K of a group, the cardinality of HK is

$$|HK| = \frac{|H||K|}{|H \cap K|}$$

Proof.

Assume we have a Tarski Monster T that is not a simple group for the sake of contradiction. There exists a normal subgroup $N \trianglelefteq T$. We find another subgroup H of T and show that $HN = NH$.

By Lemma 3.13, $HN \leq T$. By Lemma 3.14 we get that $|HN| = |H| \cdot |N| = p^2$, a contradiction. □

Acknowledgements

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Thank You!