THE RESTRICTED LIE ALGEBRA STRUCTURE ON THE BAR SPECTRAL SEQUENCE OF AN ITERATED LOOP SPACE

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ABSTRACT. There is a rich algebraic structure in the mod p homology of the iterated loop space $H_*(\Omega^n X; \mathbb{F}_p)$. It admits Lie bracket called the Browder bracket that is compatible with the Dyer-Lashof operations $Q_0, Q_1, \dots, Q_{n-1}$. Furthermore, the top Dyer-Lashof operation Q_{n-1} is a restriction for the Browder bracket. Ni proved that the Browder bracket on the homology $H_*(\Omega^n X)$ converges to the bracket on $H_*(\Omega^{n-1} X)$ in the bar spectral sequence, making it a spectral sequence of Poission-Hopf algebras. Our goal is to use the bar spectral sequence to relate the restricted Lie algebra structure given by the top Dyer-Lashof operation on $H_*(\Omega^n X; \mathbb{F}_2)$ to that of $H_*(\Omega^{n-1} X; \mathbb{F}_2)$.

1. Introduction

Algebraic topology emerged in the early 20th century as a method of constructing invariants for spaces up to homotopy. One such invariant was homology and in the mid-20th century the homology loop spaces were heavily studied by prominent figures such as William S. Browder, Frederick R. Cohen, and Eldon Dyer. We summarize their work below.

Given a space X with a chosen basepoint $*$, one can consider the associative H-space ΩX of loops in X that start and end at $*$. Iterating the loop space construction, we obtain the n-fold loop space $\Omega^n X$. The mod p homology of the n-fold loop space has a rich algebraic structure, as was studied extensively by Cohen in [5]. For instance, the graded $\mathbb{F}_2$-algebra $H_*(\Omega^n X; \mathbb{F}_2)$ is a Poisson Hopf algebra with a shifted Lie bracket called the Browder bracket that increases degree by $n-1$. It also supports Dyer-Lashof operations $Q_0, \dots, Q_{n-1}$ that are compatible with the bracket. Furthermore, the top Dyer-Lashof operation Q_{n-1} equips $H_*(\Omega^n X; \mathbb{F}_2)$ with the structure of a restricted Poisson Hopf algebra, i.e. it satisfies the adjoint identity and the Cartan formula with respect to the Browder bracket.

A natural question to ask is how the structure on $H_*(\Omega^n X; \mathbb{F}_2)$ is related to that on the homology of the delooping $B\Omega^n X \simeq \Omega^{n-1} X$ of $\Omega^n X$ for $n \geq 2$. In the limiting case $n = \infty$, Ligaard and Madsen [4] utilized the bar spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^{H_*(G)}_{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*(BG)$$

to show how the Hopf algebra structure and the Dyer-Lashof operations on the delooping BX of an infinite loop space X is determined by those on X up to extension.

In the unstable case where n is finite, Ni [7] showed that the bar spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^{H_*(\Omega^n X)}_{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*(\Omega^{n-1} X)$$

Date: January 8, 2022.

is a spectral sequence of Poisson Hopf algebras. He constructed an extension of the Browder bracket to the bar construction of $H_*(\Omega^n X)$, and showed that the extended Browder bracket passes to the Browder bracket on $H_*(\Omega^{n-1} X)$ via the bar spectral sequence.

The goal of this project is to strengthen Ni's result by taking into account the restriction on the Possion algebra $H_*(\Omega^n X; \mathbb{F}_2)$ given by the top Dyer-Lashof operation Q_{n-1}. In Section 2, we review the constructions of the Browder bracket, the Dyer-Lashof operations, the bar spectral sequence, as well as Ni's extension of the Browder bracket to the normalized bar construction $B_{*,*}(H_*(\Omega^n X))$.

In Section 3, we establish the following extension of the restriction ξ on the Browder bracket, which coincides with the top Dyer-Lashof operation Q_{n-1}, to the bar construction. This defines an operation $\xi : E^1_{s,t} \to E^1_{2s-1,2t+1}$ on the E^1-page $E^1_{s,t} = B_{s,t}(H_*(\Omega^n X))$ of the bar spectral sequence. Let $x = [x_1|\cdots|x_s] \in B_{*,*}(H_*(\Omega^n X))$. If $s = 1$, we set $\xi([x_1]) = [Q_{n-1}(x_1)]$. For $s > 1$, we take

$$\xi(x) = \sum_{\substack{(s,s)-\text{shuffles } \varphi \\ \text{with } \varphi^{-1}(1)=1}} \sum_{\substack{\varphi^{-1}(i)\leq s \\ \varphi^{-1}(i+1)>s}} [z_{\varphi^{-1}(1)}|\cdots|[z_{\varphi^{-1}(i)}, z_{\varphi^{-1}(i+1)}]|\cdots|z_{\varphi^{-1}(2s)}],$$

where

$$z_i = \begin{cases} x_i & \text{if } i \leq s \\ x_{i-s} & \text{if } i > s \end{cases},$$

and the (s,s)-shuffles are those elements of $\varphi \in \Sigma_{2s}$ satisfying $\varphi(a) < \varphi(b)$ for $1 \leq a < b \leq s$ or $s+1 \leq a < b \leq 2s$. The main result of the section is Theorem 3.4, in which we show that ξ is a restriction for the Browder bracket. That is, the bar construction is a graded restricted Lie algebra.

In Section 4, we prove that the squaring operation on the total complex of a double loop space induces an operation on the spectral sequence, making it a spectral sequence of restricted Lie algebras. We expect the induced operation to agree with the ξ operation defined above.

Understanding the restricted Lie algebra structure on the bar spectral sequence is important in many physical settings. Lie algebras are used extensively in quantum mechanics and particle physics to study the symmetries of physical systems. The structure on the homology of an iterated loop space is stronger than that of a Lie algebra, so it can provide further insight into the behavior of particles at a subatomic level.

In addition to the impact on physics, the study of Dyer-Lashof operations increases our own knowledge and understanding of algebraic topology, and in particular loop spaces. The classification of topological spaces up to homotopy equivalence was one of the main purposes of inventing the homotopy and homology groups. Computing homotopy groups is significantly harder than calculating homology and is of particular interest in the field. The loop spaces satisfy the relation $\pi_{i+1}(X) = \pi_i(\Omega X)$, so the study of loop spaces can lead to computations of homotopy groups for harder spaces.

2. Background

For a space X with basepoint $*$, we define the *loop space* ΩX as the set of functions $\gamma : I \to X$ such that $\gamma(0) = \gamma(1) = *$ equipped with the compact open topology [1]. Given

[1] It is common practice to abbreviate the unit interval $[0,1]$ as I.

two loops γ_1 and γ_2, we can define the product to be a concatenation of the loops,

$$\gamma_1 \cdot \gamma_2 = \begin{cases} \gamma_1(2t), & \text{if } 0 \leq t \leq 1/2 \\ \gamma_2(2t-1), & \text{if } 1/2 \leq t \leq 1 \end{cases}.$$

This defines a multiplication map $\Omega X \times \Omega X \to \Omega X$ that is associative up to homotopy, giving the space ΩX an associative H-space structure. This map induces a product in homology, called the *Pontryagin product*, via the maps

$$H_*(\Omega X) \times H_*(\Omega X) \xrightarrow{\otimes} H_*(\Omega X) \otimes H_*(\Omega X) \to H_*(\Omega X \times \Omega X) \to H_*(\Omega X),$$

where the second map is provided by the Künneth theorem.

The n-fold loop space is constructed inductively via $\Omega^n X = \Omega(\Omega^{n-1} X)$ and consists of the maps $\gamma : I^n \to X$ such that $\gamma(\partial I^n) = *$. It also has a multiplication

$$H_*(\Omega^n X; \mathbb{F}_2) \otimes H_*(\Omega^n X; \mathbb{F}_2) \to H_*(\Omega^n X; \mathbb{F}_2).$$

For the remainder of the paper, we suppress the coefficient $\mathbb{F}_2$.

2.1. **The structure of $H_*(\Omega^n X)$.** We construct operations on the n-fold loop space as discussed in sections 1.4.2 and 1.5.1 of [3]. Recall that $\mathcal{C}_n(k)$, the little cubes operad in I^n, is defined as the set of all rectilinear (i.e. compositions of scaling and translating) embeddings $\coprod_{j=1}^k I^n \to I^n$. There is a natural operad action on the n-fold loop space

$$\tilde{\theta} : \mathcal{C}_n(2) \times \Omega^n X \times \Omega^n X \to \Omega^n X,$$

given by "attaching" the first loop to the first cube and the second loop to the second cube.

The symmetry group Σ_2 acts on the product by swapping the order of the two cubes in $\mathcal{C}_n(2)$ and swapping the factors in $X \times X$ accordingly. Hence, $\tilde{\theta}$ factors over a map

$$\begin{tikzcd} \mathcal{C}_2(n) \times \Omega^n X \times \Omega^n X \arrow[r] \arrow[dr, "\tilde{\theta}"'] & \mathcal{C}_2(n) \times_{\Sigma_2} (\Omega^n X \times \Omega^n X) \arrow[d, "\theta"] \\ & \Omega^n X \end{tikzcd}$$

We have the following induced map in homology via the Künneth isomorphism

$$\tilde{\theta}_* : H_*(\mathcal{C}_2(n)) \otimes H_*(\Omega^n X) \otimes H_*(\Omega^n X) \xrightarrow{\text{K.I.}} H_*(\mathcal{C}_2(n) \times \Omega^n X \times \Omega^n X) \to H_*(\Omega^n X).$$

Note that $\mathcal{C}_2(n) \simeq S^{n-1}$, so we can pick a generator $\gamma \in H_{n-1}(S^{n-1}) \cong H_{n-1}(\mathcal{C}_2(n))$. The *Browder bracket* is then defined as $[x, y] = \tilde{\theta}_*(\gamma \otimes x \otimes y)$. Cohen establishes the following properties of the bracket in [5].

Theorem 2.1. *The homology $H_*(\Omega^n X)$ has a Poisson-Hopf algebra structure with a shifted Lie bracket*

$$[-,-] : H_p(\Omega^2 X) \otimes H_q(\Omega^2 X) \to H_{p+q+n-1}(\Omega^2 X),$$

known as the Browder bracket, satisfying the following relations:

- *Antisymmetry:* $[x, y] = [y, x]$;
- *Poisson Identity:* $[x, yz] = [x, y]z + y[x, z]$;
- *Jacobi Law:* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

For each n-fold loop space, there is also a *Top Dyer-Lashof operation* defined as $Q_{n-1}(x) = \theta_*(\gamma \otimes x \otimes x)$, where the induced map θ_* is

$$\theta_* : H_*(\mathcal{C}_2(n)) \otimes_{\Sigma_2} H_*(\Omega^n X) \otimes H_*(\Omega^n X) \xrightarrow{\text{K.I.}} H_*(\mathcal{C}_2(n) \times_{\Sigma_2} \Omega^n X \times \Omega^n X) \to H_*(\Omega^n X).$$

Recall that for a Lie algebra $\mathcal{A}$, a *restriction map* ξ is a map satisfying the adjoint and top additivity identities. That is,

$$\xi(x+y) = \xi(x) + \xi(y) + [x,y], \text{ and } [\xi(x), y] = [x,[x,y]],$$

which gives $\mathcal{A}$ a *restricted Lie algebra* structure. Cohen verifies these identities for the Top Dyer-Lashof operation in Section 3 of [5].

Theorem 2.2. *The Top Dyer-Lashof operation $Q_{n-1} : H_p(\Omega^n X) \to H_{2p+n-1}(\Omega^n X)$ satisfies*

- $Q_{n-1}(x+y) = Q_{n-1}(x) + Q_{n-1}(y) + [x,y]$;
- $[Q_{n-1}(x), y] = [x,[x,y]]$.

Hence, the Q_{n-1} operation is a restriction for the Browder bracket, and makes $H_*(\Omega^n X)$ a restricted Lie aglebra.

The first goal of this project is to extend the Q_{n-1} operation and the Lie algebra structure on $H_*(\Omega^n X)$ to the normalized bar construction $B_{*,*}(H_*(\Omega^n X))$.

2.2. **The normalized bar construction.** For a differential graded algebra $\mathcal{A}$ with differential $d_{\mathcal{A}}$ over a field k, we define the normalized bar construction $B_{*,*}(\mathcal{A})$ as follows. Let $\varepsilon : \mathcal{A} \to k$ be an augmentation map, and denote by $\overline{\mathcal{A}}$ the kernel of ε. Set

$$B_{s,*} = \overline{\mathcal{A}} \otimes \cdots \otimes \overline{\mathcal{A}},$$

where we repeat $\overline{\mathcal{A}}$ s times in the tensor product. In particular, $B_{0,*} = k$. The elements of $B_{s,t}$ are defined to be those elements of $B_{s,*}$ with *internal degree* t, where the internal degree of $x_1 \otimes \cdots \otimes x_s \in B_{s,*}(\mathcal{A})$ is given by $|x_1| + \cdots + |x_s|$. Typically, an element $x_1 \otimes \cdots \otimes x_s \in B_{s,*}$ is written as $[x_1|\cdots|x_s]$. Our work revolves mostly around the algebra $H_*(\Omega^n X)$, so we will further assume that $\mathcal{A}$ is a commutative $\mathbb{F}_2-$algebra.

The bar construction has an *internal differential* d of bidegree $(0,-1)$ and an *external differential* δ of bidegree $(-1,0)$ given by

$$d[x_1|\cdots|x_s] = \sum_{i=1}^{s} [x_1|\cdots|d_{\mathcal{A}} x_i|\cdots|x_s], \text{ and } \delta[x_1|\cdots|x_s] = \sum_{i=1}^{s-1} [x_1|\cdots|x_i x_{i+1}|\cdots|x_s],$$

respectively. Finally, define the *total differential* $D = d + \delta$ to be the sum of the both.

We define the comultiplication $\Delta : B_{*,*}(\mathcal{A}) \to B_{*,*}(\mathcal{A}) \otimes B_{*,*}(\mathcal{A})$ via

$$\Delta([x_1|\cdots|x_s]) = \sum_{i=0}^{s} [x_1|\cdots|x_i] \otimes [x_{i+1}|\cdots|x_s],$$

which provides a coalgebra structure (here we set $[] = 1 \in B_{0,*}$).

Definition 2.3. For two nonnegative integers p and q, a (p,q)-*shuffle* is a permutation $\varphi \in \Sigma_{[p+q]}$ satisfying $\varphi(a) < \varphi(b)$ if $1 \le a < b \le p$ or if $p+1 \le a < b \le p+q$. Note that there are $\binom{p+q}{p}$ such permutations.

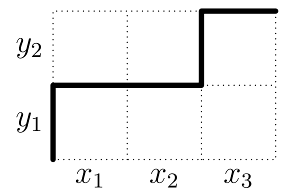
Given two elements $[x_1|\cdots|x_p]$ and $[y_1|\cdots|\alpha_q]$ in the bar construction, we define the *shuffle product* $B_{*,*}(\mathcal{A})\otimes B_{*,*}(\mathcal{A})\to B_{*,*}(\mathcal{A}\otimes\mathcal{A})$, which was first introduced in [2], to be

$$[x_1|\cdots|x_p]\otimes[y_1|\cdots|\alpha_q]\mapsto \sum_{(p,q)-\text{shuffles}}[z_{\varphi^{-1}(1)}|\cdots|z_{\varphi^{-1}(p+q)}],$$

where we set

$$z_i=\begin{cases}x_i\otimes 1 & \text{if } i\le p,\\ 1\otimes y_{i-p} & \text{if } i>p,\end{cases}.$$

Each term in the shuffle product can be realized as one of the $\binom{p+q}{p}$ paths from $(0,0)$ to (p,q) by labelling the sides of the grid and reading the variable corresponding to each segment. For instance, the path defined below from $(0,0)$ to $(2,3)$



corresponds to the term $[1\otimes y_1|x_1\otimes 1|x_2\otimes 1|1\otimes y_2|x_3\otimes 1]$ appearing in the product of $[x_1|x_2|x_3]$ and $[y_1|y_2]$.

When $\mathcal{A}$ is commutative multiplication map $\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ induces a map $B_{*,*}(\mathcal{A}\otimes\mathcal{A})\to B_{*,*}(\mathcal{A})$, which assembles into a multiplication on the bar construction $B_{*,*}(\mathcal{A})\otimes B_{*,*}(\mathcal{A})\to B_{*,*}(\mathcal{A})$ by composing with the shuffle product.

2.3. **The bar spectral sequence.** We follow the exposition provided in Section 2.2 of [7]. We define the total complex on the bar construction to be the chain complex

$$(\operatorname{tot} B_{*,*}(\mathcal{A}))_n=\bigoplus_{p+q=n}B_{p,q}(\mathcal{A})$$

with differential D (which reduces total degree by 1). The homology of this chain complex with coefficients in a field k is

$$H_*(\operatorname{tot} B_{*,*}(\mathcal{A});k)=\operatorname{Tor}_*^{\mathcal{A}}(k,k).$$

We define a filtration on each $(\operatorname{tot} B_{*,*}(\mathcal{A}))_n$ by taking

$$F_s(\operatorname{tot} B_{*,*}(\mathcal{A}))_n=\bigoplus_{\substack{p+q=n\\ p\le s}}B_{p,q}(\mathcal{A}).$$

The associated graded pieces of the filtration are

$$E^0_{s,t}=F_s(\operatorname{tot} B_{*,*}(\mathcal{A}))_{s+t}/F_{s-1}(\operatorname{tot} B_{*,*}(\mathcal{A}))_{s+t}=B_{s,t}(\mathcal{A}).$$

The differentials on the E^0 and E^1-page are given by $d_0=d$ and $d_1=\delta$. Since $\mathcal{A}$ is a k-algebra, by the Künneth isomorphism

$$H_*(B_{s,*}(\mathcal{A}))=H_*(\underbrace{\overline{\mathcal{A}}\otimes\cdots\otimes\overline{\mathcal{A}}}_{s\text{ times}})\cong\underbrace{\overline{H_*(\mathcal{A})}\otimes\cdots\otimes\overline{H_*(\mathcal{A})}}_{s\text{ times}}\cong B_{*,*}(H_*(\mathcal{A})),$$

and so the E^1-page is the bar construction of the homology of $\mathcal{A}$ with a trivial internal differential. This gives rise to a strongly convergent homological spectral sequence

$$E^2_{*,*}\cong\operatorname{Tor}_*^{H_*(\mathcal{A})}(k,k)\Rightarrow\operatorname{Tor}_*^{\mathcal{A}}(k,k). \tag{2.1}$$

Here we are interested in the case $\mathcal{A} = C_*(\Omega^{n-1}X)$. It is well known that there is a quasi-isomorphism

$$\operatorname{tot} B_{*,*}(C_*(\Omega^n X)) \xrightarrow{\simeq} C_*(\Omega^{n-1}X).$$

Passing to homology yields an isomorphism of Hopf algebras

$$\operatorname{Tor}_*^{C_*(\Omega^n X)}(k,k) \cong H_*(\Omega^{n-1}X).$$

Clark [1] proves that 2.1 is a spectral sequence of Hopf algebras. To sum up, the spectral sequence relates the bar construction of $H_*(\Omega^n X)$ to the homology of $\Omega^{n-1}X$.

2.4. **The Browder bracket in the bar construction.** The bar construction $B_{*,*}(\mathcal{A})$ inherits a Hopf algebra structure as mentioned in Section 1. In [7], Ni proves that if $\mathcal{A}$ is a commutative Poisson DGA, then $B_{*,*}(\mathcal{A})$ inherits a commutative differential bigraded Poission-Hopf algebra structure. We reformulate his main results for our purposes here.

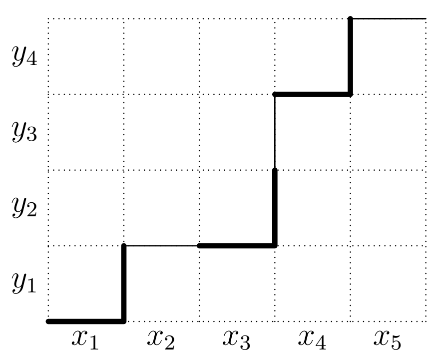
Definition 2.4. [7, Theorem 3.1] If $\mathcal{A}$ is a commutative Poisson $\mathbb{F}_2$ differential graded algebra with bracket of degree $n-1$, then the bracket on $B_{*,*}(\mathcal{A})$ has bidegree $(-1, n-1)$, and is constructed by defining $[[x_1|\cdots|x_p],[y_1|\cdots|y_q]]$ to be

$$\sum_{(p,q)-\text{shuffles }\varphi} \sum_{\substack{\varphi^{-1}(i)\leq p\\ \varphi^{-1}(i+1)>p}} [z_{\varphi^{-1}(1)}|\cdots|[z_{\varphi^{-1}(i)}, z_{\varphi^{-1}(i+1)}]|\cdots|z_{\varphi^{-1}(p+q)}],$$

where

$$z_i = \begin{cases} x_i & \text{if } i \leq p \\ y_{i-p} & \text{if } i > p \end{cases}.$$

Visually, the bracket term $[a_{\varphi^{-1}(i)}, a_{\varphi^{-1}(i+1)}]$ can be visualized as a joint along a path from $(0,0)$ to (p,q). Here, we define a joint along a path to be two adjacent steps with the first oriented horizontally and the second oriented vertically. The joints along one possible path from $(0,0)$ to $(6,4)$ are emboldened below.



In the expansion of $[[x_1|\cdots|x_6],[y_1|\cdots|y_4]]$, one can place a bracket along any of the emboldened joints in the path. This path, in particular, yields

$$[[x_1,y_1]|x_2|x_3|y_2|y_3|x_4|y_4|x_5] + [x_1|y_1|x_2|[x_3,y_2]|y_3|x_4|y_4|x_5] + [x_1|y_1|x_2|x_3|y_2|y_3|[x_4,y_4]|x_5].$$

Ni also proves that the bracket satisfies the following expected compatibility relations with the differential and product.

Proposition 2.5. *The bracket defined in Definition 2.2 satisfies the following identities:*

- $[x,y] = [y,x]$*;*
- $[x,yz] = [x,y]z + y[x,z]$*;*

- $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$;
- $\delta[x,y] = [\delta x, y] + [x, \delta y]$ *and* $d[x,y] = [dx,y] + [x,dy]$,

which make $B_{*,*}(\mathcal{A})$ *a commutative differential bigraded Poisson Hopf algebra satisfying the Liebniz rule with respect to both derivatives.*

3. The ξ Operation in the Bar Construction

Ni proves in [7] that the bar construction $B_{*,*}(\mathcal{A})$ is a commutative Hopf algebra with a Lie bracket extending the bracket on $\mathcal{A}$ for a commutative Hopf DGA over $\mathbb{F}_2$. Our goal is to extend the restriction on $\mathcal{A}$ to a restriction on the bar construction for the Lie bracket Ni constructs, and apply our results to the E^1-term of the spectral sequence.

Let $\mathcal{A}$ be a commutative augmented DGA over $\mathbb{F}_2$ with a restricted Lie algebra structure. Suppose the restriction $\psi : \mathcal{A} \to \mathcal{A}$ induces a restricted Lie algebra structure on the augmentation ideal $\overline{\mathcal{A}}$. We introduce here an operation ξ on the bar construction of $\mathcal{A}$, which equips the normalized bar construction $B_{*,*}(A)$ with a restricted Lie algebra structure.

Construction. Let $x = [x_1|\cdots|x_s]$. If $s=1$, we set $\xi(x) = [\psi(x_1)]$. For $s>1$, we take

$$\xi(x) = \sum_{\substack{(s,s)-\text{shuffles } \varphi \\ \text{with } \varphi^{-1}(1)=1}} \sum_{\substack{\varphi^{-1}(i)\le s \\ \varphi^{-1}(i+1)>s}} [z_{\varphi^{-1}(1)}|\cdots|[z_{\varphi^{-1}(i)}, z_{\varphi^{-1}(i+1)}]|\cdots|z_{\varphi^{-1}(2s)}],$$

where

$$z_i = \begin{cases} x_i & \text{if } i \le s \\ x_{i-s} & \text{if } i > s \end{cases}.$$

We extend ξ to the entire bar construction via top additivity,

$$\xi(x+y) = \xi(x) + \xi(y) + [x,y],$$

for any x and y.

The construction takes only the paths from $(0,0)$ to (s,s) in the expansion of the self-bracket $[x,x]$ that begin with a horizontal step. In this sense, ξ can be thought of as "half the bracket of $[x,x]$" since exactly half the paths from $(0,0)$ to (s,s) begin with a horizontal step.

First, we must verify that ξ is a well-defined operation.

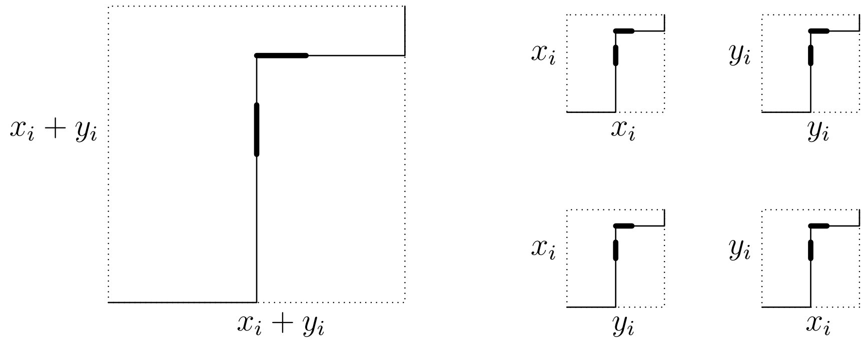
Proposition 3.1. *The operation* ξ *is a well-defined on* $B_{*,*}(\mathcal{A})$.

Proof. To check ξ is well defined, we must check ξ is well defined on $B_{s,*}$. It is straightforward to check that $\xi(cx) = c\xi(x)$ for any $c \in \mathbb{F}_2$. Next, we must check that if $x = [x_1|\cdots|x_i|\cdots|x_s]$ and $y = [y_1|\cdots|y_i|\cdots|y_s]$, then $\xi(x+y) = \xi(x) + \xi(y) + [x,y]$, provided that $x_j = y_j$ for all $j \neq i$. That is, we must verify top additivity in the case where the two tensors may differ only at the ith entry.

If $s=1$, the result follows from

$$\xi([x_1]+[y_1]) = [\psi(x_1)] + [\psi(y_1)] + [[x_1,y_1]] = [\psi(x_1+y_1)] = \xi([x_1+y_1]).$$

For $s>1$, refer to the terms in $\xi(x+y)$ as paths from $(0,0)$ to (s,s). Since the bracket is bilinear, each path can be broken into four components as shown below:



The top two paths contribute to $\xi(x)$ and $\xi(y)$, respectively, while the bottom two paths contribute to $[x, y]$ (this can be seen after reflecting one of the diagrams across the diagonal). ■

Our next step is to provide a simplified formula for ξ that allows for simpler computation.

Definition 3.2. Call an (s, s)-shuffle *disposable* if there is an i such that $\varphi^{-1}(2i) \in \{i+s, i\}$. Otherwise, call it *indisposable*.

Visually, the disposable terms are those that cross the diagonal connecting $(0, 0)$ to (s, s). The claim is that we can effectively ignore the disposable terms appearing in the definition of ξ.

Proposition 3.3. *For $x = [x_1|\cdots|x_s]$ and $s > 1$, we have*

$$\xi(x) = \sum_{\substack{\text{indisposable } (s,s)-\text{shuffles } \varphi \\ \text{with } \varphi^{-1}(1)=1}} \sum_{\substack{\varphi^{-1}(i)\leq s \\ \varphi^{-1}(i+1)>s}} [z_{\varphi^{-1}(1)}|\cdots|[z_{\varphi^{-1}(i)}, z_{\varphi^{-1}(i+1)}]|\cdots|z_{\varphi^{-1}(2s)}],$$

where the z_i's are defined in the same way as before.

In other words, the restriction map ξ only traverses the paths from $(0, 0)$ to (s, s) that lie strictly below the diagonal. In other words, it traverses the Catalan paths.

Proof. We must show that the sum of all the disposable terms evaluates to 0. Let P be a disposable path, and suppose it first crosses the diagonal at (i, i). It suffices to prove that the sum S of all disposable paths that agree with P for the first $2i$ steps is 0 (since if they agree for the first $2i$ steps then the path must have hit the diagonal at (i, i) first).

Let R_{2i} denote the first $2i$ steps of P. Note that the bracket appearing along the path can appear after (i, i) or before (i, i). This yields

$$\mathcal{S} = R_{2i} \otimes [[x_{i+1}|\cdots|x_s], [x_{i+1}|\cdots|x_s]] + R_{2i} \otimes ([x_{i+1}|\cdots|x_s] \cdot [x_{i+1}|\cdots|x_s]) = 0,$$

and we are done. ■

Proposition 3.3 allows for simpler computations of the ξ operation.

Example. The value of ξ on two terms is given by

$$\xi([x|y]) = [x|[x, y]|y],$$

which follows from noting there is only one path that falls under the diagonal from $(0, 0)$ to $(2, 2)$.

The main result of this section is Theorem 3.4, which verifies that ξ is a restriction for the bracket on $B_{*,*}(\mathcal{A})$.

Theorem 3.4. *If $\mathcal{A}$ is a commutative restricted Poisson $\mathbb{F}_2$-DGA with a restriction ψ that can be restricted to the augmentation ideal $\overline{A}$, the operation $\xi : B_{s,t}(\mathcal{A}) \to B_{2s-1.2t+1}(\mathcal{A})$ is a restriction for the bracket:*

- $\xi(x+y) = \xi(x) + \xi(y) + [x,y]$;
- $[x, \xi y] = [y, [x,y]]$;
- $\delta\xi x = [x, \delta x]$.
- *Furthermore, if $d_{\mathcal{A}}\psi x = [x, d_{\mathcal{A}} x]$ holds in $\mathcal{A}$, then $dx = [x, dx]$.*

In particular, the bar construction $B_{*,*}(\mathcal{A})$ is a restricted Lie algebra.

Proof. Many of the proofs have similar combinatorial taste, so we do not include full proofs for all. It's easy to check that degree-wise all the identities are correct.

- The result readily follows from Proposition 3.3.
- It suffices to check the identity only on the elementary tensors. Indeed, if x and z are pure tensors, then

$$\begin{aligned}[\xi(x+z), y] &= [\xi(x) + \xi(z) + [x,z], y] \\ &= [\xi(x), y] + [\xi(z), y] + [[x,z], y] \\ &= [x,[x,y]] + [z,[z,y]] + [[x,z],y] \\ &= [x,[x,y]] + [z,[z,y]] + [z,[x,y]] + [x,[z,y]] \\ &= [x+z, [x+z, y]],\end{aligned}$$

by the Jacobi Law. By induction, we find that $[\xi(x), y] = [x, [x,y]]$ for all x. The same argument holds for y, since

$$\begin{aligned}[\xi(x), y+z] &= [\xi(x), y] + [\xi(x), z] \\ &= [x,[x,y]] + [x,[x,z]] \\ &= [x, [x,y] + [x,z]] \\ &= [x, [x, y+z]],\end{aligned}$$

and induction once again extends the identity to all y. We now proceed to establish the result when x and y are pure tensors.

Let $x = [x_1|\cdots|x_p]$ and $y = [y_1|\cdots|y_q]$. The case $p = 1$ must be handled separately. It suffices to show

$$[\xi(x), y] = [[\psi(x_1)], [y_1|\cdots|y_q]] = [[x_1], [[x_1], [y_1|\cdots|y_q]]] = [x, [x,y]].$$

By bilinearity and the adjoint identity,

$$\begin{aligned}
[[x_1],[[x_1],[y_1|\cdots|y_q]]] &= \left[[x_1],\sum_{i=1}^{q}[y_1|\cdots|[x_1,y_i]|\cdots|y_q]\right]\\
&= \sum_{i=1}^{q}[[x_1],[y_1|\cdots|[x_1,y_i]|\cdots|y_q]]\\
&= 2\sum_{1\le i<j\le q}[y_1|\cdots|[x_1,y_i]|\cdots|[x_1,y_j]|\cdots|y_q]\\
&\quad + \sum_{i=1}^{q}[y_1|\cdots|[x_1,[x_1,y_i]]|\cdots|y_q]\\
&= \sum_{i=1}^{q}[y_1|\cdots|[x_1,[x_1,y_i]]|\cdots|y_q]\\
&= \sum_{i=1}^{q}[y_1|\cdots|[\psi(x_1),y_i]|\cdots|y_q]\\
&= [[\psi(x_1)],[y_1|\cdots|y_q]].
\end{aligned}$$

The case where $p>1$ succumbs to an elementary combinatorial argument, which we omit here.

- If x and y are pure tensors,

$$\begin{aligned}
\delta\xi(x+y) &= \delta(\xi(x)+\xi(y)+[x,y])\\
&= \delta\xi(x)+\delta\xi(y)+\delta[x,y]\\
&= [x,\delta x]+[y,\delta y]+[y,\delta x]+[x,\delta y]\\
&= [x+y,\delta(x+y)].
\end{aligned}$$

By induction it follows that we need only establish the result for a pure tensor. Let $x=[x_1|\cdots|x_s]$. If $s=1$, we have

$$\delta\xi(x)=\delta[\psi(x_1)]=0=[x,0]=[x,\delta x].$$

Now we must establish the result for $s>1$.

For simplicity, we introduce the operator

$$\delta_i([x_1|\cdots|x_s])=[x_1|\cdots|x_ix_{i+1}|\cdots|x_s],$$

for $1\le i<s$ and extend linearly. For $i\ge s$, we simply set $\delta_i(x)=0$. Note that $\delta(x)=\sum_{i=1}^{s-1}D_i(x)$. Thus, it suffices to prove

$$\sum_{k=1}^{2s-2}\delta_k\xi(x)=\sum_{j=1}^{s-1}[x,\delta_jx].$$

First, we need to start by unpacking $\xi(x)$ visually. We shall refer to the terms in $\xi(x)$ as paths. Suppose there is a path in $\xi(x)$ with a bracket at $[x_i,x_j]$ (we assume that this bracket is not $[x_1,x_1]=0$, since those terms vanish). We can represent that term visually as a path P from $(0,0)$ to (s,s) with a hinge at $[x_i,x_j]$ starting off with a horizontal step.

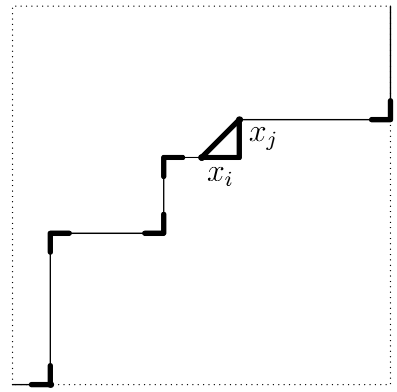


Figure 3.1

Define a switch to be a place where direction changes, and embolden them. We investigate what occurs upon taking the differential across all paths P in $\xi(x)$.

Select a $1 \leq k \leq 2s-2$ and consider the operator δ_k. Each δ_k multiplies adjacent steps along P. We consider where these adjacent multiplications take place.

– The operator δ_k could multiply two terms adjacent to a switch (indicated by red). Suppose the path is of the form $[x_1|\cdots|x_a|x_b|\cdots|[x_i,x_j]|\cdots]$ where there is a switch occurring between x_a and x_b. By swapping terms at the switch, we find that
$[x_1|\cdots|x_b|x_a|\cdots|[x_i,x_j]|\cdots]$ is also a path in $\xi(x)$. Taking $k = a+b-1$, we see

$$\begin{aligned}&\delta_k([x_1|\cdots|x_a|x_b|\cdots|[x_i,x_j]|\cdots]) + \delta_k([x_1|\cdots|x_b|x_a|\cdots|[x_i,x_j]|\cdots])\\ &= 2[x_1|\cdots|x_a x_b|\cdots|[x_i,x_j]|\cdots]\\ &= 0.\end{aligned}$$

Hence, for each path P and a given k, we can find a conjugate path P' such that $\delta_k(P) + \delta_k(P') = 0$. Therefore, we can effectively ignore all the switches along the paths in $\xi(x)$.

– Alternatively, δ_k could multiply two terms in the same direction. In this case, the differential multiplies x_ℓ with $x_{\ell+1}$ for some ℓ. From the observation above, these steps must be both oriented vertically or oriented horizontally. The key idea is to consider all terms in the expansion of $\delta\xi(x)$ that contain the step $x_\ell x_{\ell+1}$ and show each term has a corresponding term in the expansion of $[x, \delta_\ell x]$ (this is of course, excluding the terms where the hinge is at $x_\ell x_{\ell+1}$).

Fortunately this turns out to be a rather simple combinatorial argument. Let P be a path such that x_ℓ and $x_{\ell+1}$ appear as consecutive steps (with the same orientation). Figure 3.2 shows all the possible segments (x_ℓ followed by $x_{\ell+1}$) that P could pass through with the possible horizontal and vertical segments dashed. Note that δ simply collapses the two steps x_ℓ and $x_{\ell+1}$ into the single step $x_\ell x_{\ell+1}$.

We consider first, paths P which go through the horizontal segments. In this case, we can collapse the steps x_ℓ and $x_{\ell+1}$ into the single horizontal step $x_\ell x_{\ell+1}$ as shown in the diagram to the right in Figure 3.2. But once we do this, P can simply be viewed as a term in $[x, \delta_\ell x]$. In fact, P corresponds to one of the terms in $[x, \delta_\ell x]$ in which the steps x_1 and $x_\ell x_{\ell+1}$ have the same orientation.

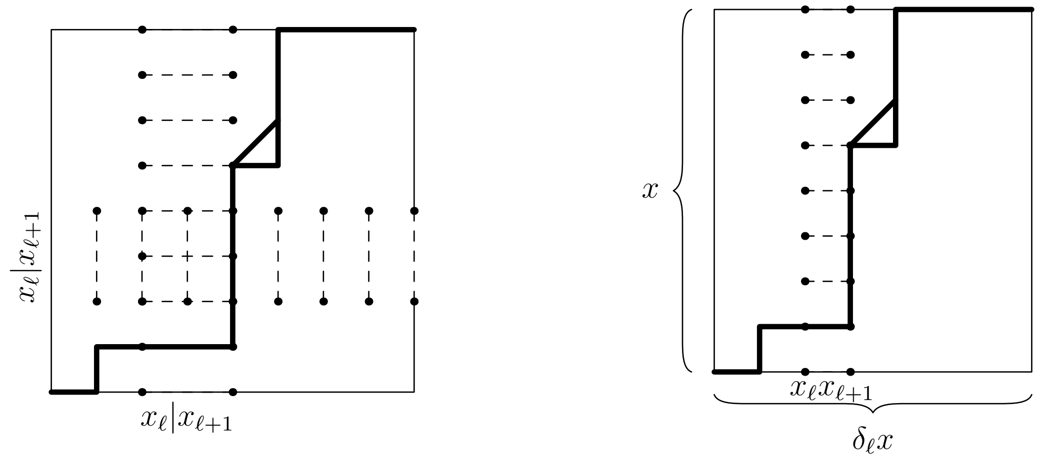
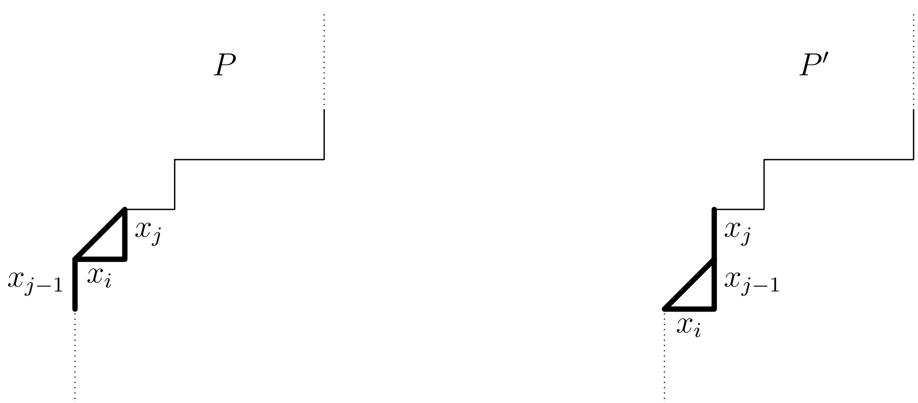


Figure 3.2

If P passes through the vertical segments, essentially the same argument applies. This time, the differential δ collapses the vertical steps x_ℓ and $x_{\ell+1}$ into the single vertical step $x_\ell x_{\ell+1}$. However, the resulting path instead corresponds to the terms in $[x, \delta_\ell x]$ in which the steps x_1 and $x_\ell x_{\ell+1}$ have opposite orientation.

– The last possibility is that δ_k multiplies terms adjacent to the hinge. As in Figure 3.1, suppose there is a bracket at $[x_i, x_j]$. Then, the step before the hinge must either be x_{i-1} or x_{j-1} and the step after the hinge must be either x_{i+1} or x_{j+1}. The differential either multiplies the first step (that is x_{i-1} or x_{j-1}) with the hinge, or it multiplies the second step (x_{i+1} or x_{j+1}) with the hinge.

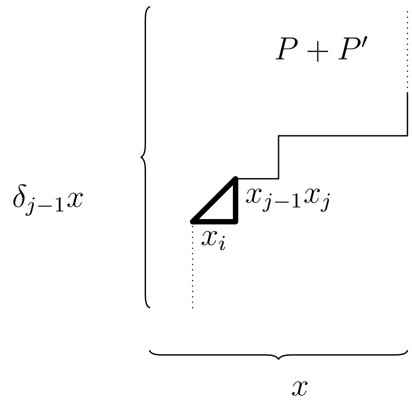
Without loss of generality, assume that the differential multiplies x_{i-1} with the hinge. The other cases are similar. Then, the path P in the expansion of $\delta\xi(x)$ is of the form $[x_1|\cdots|x_{j-1}|[x_i, x_j]|\cdots]$.



We can find a path P' with exactly the same steps, only differing at the $(i+j-2)$ and $(i+j-1)$th step given by $P' = [x_1|\cdots|[x_{j-1}, x_i]|x_j]|\cdots]$, as shown above. But this gives

$$\begin{aligned}
\delta_{i+j-2}P + \delta_{i+j-2}P' &= [x_1|\cdots|x_{j-1}[x_i, x_j]|\cdots] + [x_1|\cdots|[x_{j-1}, x_i]x_j]|\cdots] \\
&= [x_1|\cdots|x_{j-1}[x_i, x_j] + [x_{j-1}, x_i]x_j]|\cdots] \\
&= [x_1|\cdots|[x_i, x_{j-1}x_j]|\cdots],
\end{aligned}$$

by the Poisson identity. But this final term corresponds to a path in $[x, \delta_{j-1}x]$, with the same opening and ending steps as P and P'. If instead, the differential multiplied the hinge with the step following it, $P + P'$ would correspond to a path in $[x, \delta_j x]$.



Hence, the terms where the differential multiplies the hing with a neighboring step are also accounted for in $[x, \delta x]$.

- By a similar argument, we must prove the second identity for pure tensors only. If $s = 1$, then the identity holds under the assumption that $d_{\mathcal{A}}\psi x = [x, d_{\mathcal{A}}x]$. For the remainder of the proof, let $s > 1$.

  We begin by analyzing the left-hand-side $d\xi x$. As before, we introduce the operator

$$d_i[x_1|\cdots|x_s] = [x_1|\cdots|d_{\mathcal{A}}x_i|\cdots|x_s],$$

for $1 \leq i \leq s$, and 0 otherwise. Under this notation, we must prove

$$\sum_{k=1}^{2s} d_k\xi(x) = \sum_{j=1}^{s}[x, d_j x].$$

Suppose $\xi(x)$ contains a term of the form $[x_1|\cdots|[x_i, x_j]|\cdots]$. The differential can apply $d_{\mathcal{A}}$ to one of three emboldened locations as shown in Figure 3.3.

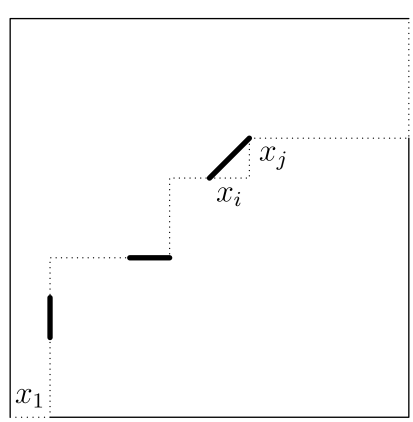


Figure 3.3

– The differential could apply $d_{\mathcal{A}}$ to the bracket (i.e. the $(i+j-1)$th term in the tensor). By Leibniz,

$$\begin{aligned} d_{i+j-1}[x_1|\cdots|[x_i, x_j]|\cdots] &= [x_1|\cdots|d_{\mathcal{A}}[x_i, x_j]|\cdots] \\ &= [x_1|\cdots|[dx_i, x_j]|\cdots] + [x_1|\cdots|[x_i, dx_j]|\cdots]. \end{aligned}$$

The former term appears once in $[x, d_i x]$ and the latter once in $[x, d_j x]$. The former term can be realized as a path in the expansion of $[x, d_i x]$ after reflecting across the main diagonal (the reflection must be done since the first step will always be horizontal in ξ).

– Alternatively, the differential could apply $d_{\mathcal{A}}$ to a vertical step, say at x_i, along the path. In this case, the path can immediately be interpreted as a path in $[x, d_i x]$. Finally, if the differential is applied to a horizontal step along the path, we can reflect it over the main diagonal as in the argument above.

Hence, the paths appearing in $d\xi x$ and $[x, dx]$ can be placed in a bijection with each other. This establishes the final identity, and we are done.

■

The culmination of this section is that the E^1 term in our spectral sequence is a restricted Lie algebra.

Corollary 3.5. *The E^1-page of the spectral sequence given by $B_{*,*}(H_*(\Omega^n X))$ is a restricted Lie algebra.*

Proof. Let $*$ denote the space of a single point. There is a natural induced augmentation in homology $H_*(\Omega^n X) \to H_*(*) \cong \mathbb{F}_2$. It is straightforward to check that the Q_{n-1} operation may be restricted to $\overline{H_*(\Omega^n X)}$. Furthermore, the differential on homology is trivial, so the ξ operator is compatible with the internal differential and is a restriction for the Browder bracket on the E^1 page. ■

4. Structure on the Bar Spectral Sequence

Our goal is to prove that the bar spectral sequence of $C_*(\Omega^n X)$ is a spectral sequence of restricted Lie algebras by defining an operation on the total complex that induces a restriction on the spectral sequence. We expect the induced operation on the E^1-page to be identical to the ξ operation constructed in Section 3.

4.1. **A commutative multiplication on the chain complex.** We follow here the exposition provided in [7]. The multiplication map defined in Section 2 relies on the commutativity of the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ (otherwise, there is no algebra homomorphism $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$). In the case $\mathcal{A} = C_*(\Omega G)$, the multiplication is not commutative. However (c.f. [1]), it is still possible to define a multiplication on the spectral sequence that matches with the shuffle product on the E^0 and E^1 page.

Note that Ω is a functor from the category of pointed spaces to itself, so there is an induced map

$$M_0 : \Omega G \times \Omega G \cong \Omega(G \times G) \to \Omega G,$$

sending (γ_1, γ_2) to the pointwise product $\gamma_1(t)\gamma_2(t)$ for $t \in I$. Unlike [7], since we have used the non-associative model, the two spaces $\Omega G \times \Omega G$ and $\Omega(G \times G)$ are identical.

For the map M_0, Clark [1] constructs homotopies

$$M_n : (\Omega G \times \Omega G) \times (I \times \Omega G \times \Omega G)^n \to \Omega G$$

satisfying

$$M_n(y_0, t_1, y_1, \cdots, t_n, y_n) = \begin{cases} M_{n-1}(y_0, t_1, \cdots, t_{i-1}, y_{i-1}y_i, t_{i+1}, \cdots, t_n, y_n) & \text{if } t_i = 0 \\ M_0(y_0, t_1, \cdots, t_{i-1}, y_{i-1})M_{n-i}(y_i, t_{i+1}, \cdots, t_n, y_n) & \text{if } t_i = 1 \end{cases}.$$

Lemma 2.2 of Sugawara [8] constructs a delooping $B(\Omega G \times \Omega G) \to B\Omega G$ from the M_n's. Each homotopy M_n induces the chain map

$$(C_*(\Omega G) \otimes C_*(\Omega G)) \otimes (C_*(I) \otimes C_*(\Omega G) \otimes C_*(\Omega G))^n \to C_*(\Omega G).$$

Taking the identity map in $C_1(I)$ for each occurrence yields a map

$$h_n : (C_*(\Omega G) \otimes C_*(\Omega G))^{n+1} \to C_*(\Omega G).$$

We obtain the algebra homomorphism from $B_{*,*}(\mathcal{A} \otimes \mathcal{A}) \to B_{*,*}(\mathcal{A})$ by mapping

$$\begin{aligned} &[x_1 \otimes y_1| \cdots |x_m \otimes y_m] \\ &\quad \mapsto \sum_{i_1+\cdots+i_k=m} [h_{i_1-1}(x_1 \otimes y_1| \cdots |x_{i_1} \otimes y_{i_1})| \cdots |h_{i_k-1}(x_{m-i_k+1} \otimes y_{m-i_k+1}| \cdots |x_m \otimes y_m)]. \end{aligned}$$

This defines a multiplication on the total complex after composing with the shuffle product,

$$B_{p,q}(C_*(\Omega G)) \otimes B_{s,t}(C_*(\Omega G)) \to B_{p+s,q+t}(C_*(\Omega G) \otimes C_*(\Omega G)) \to F_{p+s}(\text{tot } B_{*,*}(C_*(\Omega G))),$$

and extending linearly.

4.2. **The double loop space.** We begin by studying the spectral sequence for the double loop space. Ni [7] proves that the commutator $[x, y] = xy + yx$ on the total complex induces a bracket on the spectral sequence making it a spectral sequence of Poission Hopf algebras. He also shows that the commutator converges to the commutator on $H_*(\Omega X)$. Our goal is to show that the squaring operation x^2 induces a restriction Lie algebra structure on the spectral sequence that converges to the squaring operation on $H_*(\Omega X)$.

Proposition 4.1. *The bar spectral sequence*

$$E^2_{s,t} = \text{Tor}^{H_*(\Omega^2 X)}_{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*(\Omega X)$$

is a spectral sequence of restricted Lie algebras, with a restriction $\zeta_r : E^r_{s,t} \to E^r_{2s-1,2t+1}$ *satisfying* $d_r\zeta_r x = [x, d_r x]$, *where* $[-,-] : E^r_{s,t} \otimes E^r_{p,q} \to E^r_{p+s-1,q+t+1}$ *is the bracket induced by the commutator.*

Proof. Let $J = \text{tot}\, B_{*,*}(H_*(\Omega^2 X))$ denote the total complex and let $\zeta(x) = x^2$ be the squaring operation. By Leibniz,

$$D\zeta(x) = xDx + (Dx)x = [x, Dx],$$

where $[a, b] = ab + ba$ denotes the commutator. Ni establishes that the commutator defines a map $F_pJ_s \otimes F_qJ_t \to F_{p+q-1}J_{s+t}$ ([7]). Fix an $x \in F_pJ_s$. Since the h_0 map is multiplication, the terms appearing in filtration F_{2p} cancel. Thus, ζ defines a map $F_pJ_s \to F_{2p-1}J_{2s}$.

Define

$$\begin{aligned} Z^r_{p,q} &= F_pJ_{p+q} \cap D^{-1}(F_{p-r}J_{p+q-1}), \\ B^r_{p,q} &= F_pJ_{p+q} \cap D(F_{p+r}J_{p+q+1}). \end{aligned}$$

Then, the rth page is given by

$$E_r^{p,q} = Z_{p,q}^r/(Z_{p+1,q-1}^{r+1} + B_{p,q}^{r+1}),$$

as in [6].

The first step is to check that ζ defines a map $Z_{p,q}^r \to Z_{2p-1,2q+1}^r$. Take

$$x \in Z_{p,q}^r = F_p J_{p+q} \cap D^{-1}(F_{p-r}J_{p+q-1}).$$

Note $\zeta(x) \in F_{2p-1}J_{2p+2q}$ and $Dx \in F_{p-r}J_{p+q-1}$. By the Leibniz relation,

$$D\zeta(x) = [x, Dx] \in F_{2p-1-r}J_{2p+2q-1},$$

and so $\zeta(x) \in D^{-1}(F_{2p-1-r}J_{2p+2q-1})$. We conclude that

$$\zeta(x) \in F_{2p-1}J_{2p+2q} \cap D^{-1}(F_{2p-1-r}J_{2p+2q-1}) = Z_{2p-1,2q+1}^r.$$

The next step is to verify that

$$\zeta(Z_{p+1,q-1}^{r+1} + B_{p,q}^{r+1}) \subseteq Z_{2p.2q}^{r+1} + B_{2p-1,2q+1}^{r+1}.$$

Take $a \in Z_{p+1,q-1}^{r+1}$ and $b \in B_{p,q}^{r+1}$, and note

$$\zeta(a+b) = \zeta(a) + \zeta(b) + [a,b].$$

Since $D\zeta(b) = 0$, we have $\zeta(b) \in D^{-1}(F_{2p-r-1}J_{2p+2q+1})$. However, because $b \in F_pJ_{p+q}$, we have $\zeta(b) \in F_{2p-1}J_{2p+2q}$. It follows that

$$\zeta(b) \in F_{2p}J_{2p+2q} \cap D^{-1}(F_{2p-r-1}J_{2p+2q+1}) \subseteq Z_{2p,2q}^{r+1}.$$

Now we analyze $\zeta(a)$. Note that $a \in Z_{p,q}^r$, and so $a \in F_pJ_{p+q}$. This means that $\zeta(a) \in F_{2p-1}J_{2p+2q}$. Note further that since $a \in Z_{p+1,q-1}^{r+1}$, that $Da \in F_{p-r}J_{p+q-1}$. Next, we see that

$$D\zeta(a) = [a, Da] \in F_{2p-r-1}J_{2p+2q-1},$$

and so we conclude that $\zeta(a) \in F_{2p-1}J_{2p+2q} \cap D^{-1}(F_{2p-r-1}J_{2p+2q-1}) \subseteq Z_{2p,2q}^{r+1}$. Finally, we must handle the bracket. This is easy since,

$$D[a,b] = D[a, Dc] = [Da, Dc] = [Da, b] \in F_{2p-r-1}J_{2p+2q-1}.$$

However, since both a and b are in F_pJ_{p+q} (as they are both in $Z_{p,q}^r$) we also have $[a,b] \in F_{2p-1}J_{2p+2q}$. Thus, the bracket is contained in $Z_{2p,2q}^{r+1}$ and we obtain stronger statement

$$\zeta(Z_{p+1,q-1}^{r+1} + B_{p,q}^{r+1}) \subseteq Z_{2p,2q}^{r+1}.$$

Hence, ζ induces an operation $\zeta_r : E_{p,q}^r \to E_{2p-1,2q+1}^r$ such that the following diagram commutes:

$$\begin{tikzcd}
Z_{p,q}^r \arrow[r, "\zeta"] \arrow[d] & Z_{2p-1,2q+1}^r \arrow[d] \\
E_{p,q}^r \arrow[r, "\zeta_r"'] & E_{2p-1,2q+1}^r
\end{tikzcd}$$

The vertical maps are the canonical projection maps $Z_r^{p,q} \to E_r^{p,q}$.

It remains to establish that ξ_r, the induced ζ operation on the E^r-page, satisfies $d_r\xi_r x = [x, d_r x]$. This follows readily from the commutativity of the following diagrams:

$$\begin{tikzcd}
Z^r_{p,q} \arrow[r, "\zeta"] \arrow[d] & Z^r_{2p-1,2q+1} \arrow[r, "D"] \arrow[d] & Z^r_{2p-1-r,2q+r} \arrow[d] \\
E^r_{p,q} \arrow[r, "\zeta_r"'] & E^r_{2p-1,2q+1} \arrow[r, "d_r"'] & E^r_{2p-1-r,2q+r}
\end{tikzcd}$$

$$\begin{tikzcd}
Z^r_{p,q} \arrow[r, "x\mapsto x\otimes Dx"] \arrow[d] & Z^r_{p,q} \otimes Z^r_{p-r,q+r-1} \arrow[r, "a\otimes b\mapsto [a,b]"] \arrow[d] & Z^r_{2p-1-r,2q+r} \arrow[d] \\
E^r_{p,q} \arrow[r, "x\mapsto x\otimes d_r x"'] & E^r_{p,q} \otimes E^r_{p-r,q+r-1} \arrow[r, "a\otimes b\mapsto [a,b]"'] & E^r_{2p-1-r,2q+r}
\end{tikzcd}$$

Since $D\zeta(x) = [x, Dx]$, the identity holds true on $E^r_{*,*}$. Similarly, we find:

- The operator ξ_r satisfies the adjoint identity, since ζ satisfies the adjoint identity.
- The operator ξ_r satisfies top additivity, since ζ satisfies top additivity.

■

Conjecture 4.2. *The operation ζ_1 on the E^1-page of the spectral sequence agrees with the ξ operation on $B_{*,*}(H_*(\Omega^2 X))$.*

5. Acknowledgements

I am extremely grateful to my mentor Adela Zhang for helping me throughout the entire research process and providing me with the encouragement to continue trying to find results even during a time when I felt there were none I could find, and for also providing lots of feedback on several drafts of my paper. I would also like to thank PRIMES for offering me this unique opportunity to conduct mathematics research. Finally, I would like to thank Prof. Haynes Miller for proposing the project and shaping its course.

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