MEROMORPHIC FUNCTIONS WITH THE SAME PREIMAGES AT SEVERAL FINITE SETS

KENTA SUZUKI AND MICHAEL E. ZIEVE

Abstract. Let p and q be nonconstant meromorphic functions on $\mathbb{C}^m$. We show that if p and q have the same preimages as one another, counting multiplicities, at each of four nonempty pairwise disjoint finite subsets $S_1, \dots, S_4$ of $\mathbb{C}$, then p and q have the same preimages as one another at each of infinitely many finite subsets of $\mathbb{C}$, and moreover $g(p) = g(q)$ for some nonconstant rational function $g(x)$ whose degree is bounded in terms of the sizes of the S_i's. This result is new already when $m = 1$, and it implies many previous results about the extent to which a meromorphic function is determined by its preimages of a few points or a few small sets.

1. Introduction

As a consequence of his theory of value distribution of meromorphic functions, Nevanlinna [27] showed that a nonconstant meromorphic function on the complex plane is uniquely determined by its inverse images at any five points of the Riemann sphere $\mathbb{C}_\infty$. He also showed that if nonconstant meromorphic functions p, q on the complex plane have the same preimages as one another, counting multiplicities, at each of four points in $\mathbb{C}_\infty$, then there is a Möbius transformation μ such that $p = \mu \circ q$. In this paper we develop a new theory which addresses preimages of sets rather than merely preimages of points. In case the sets have size 1, our results generalize Nevanlinna's four-values theorem and the "counting multiplicities" version of Nevanlinna's five-values theorem. We will use the following standard terminology:

Notation. We write $\mathcal{M}(M)$ for the set of meromorphic functions on a complex manifold M (which in this paper can always be assumed to be either $\mathbb{C}^m$ or a compact Riemann surface such as the Riemann sphere $\mathbb{C}_\infty$).

Definition 1.1. We say that $p, q \in \mathcal{M}(M)$ share CM a subset S of $\mathbb{C}_\infty$ if the p-preimages of S coincide with the q-preimages of S, counting multiplicities.

Definition 1.1 involves the multiplicity of an element of M under an element of $\mathcal{M}(M)$. We will recall the definition of this notion in Section 2. We note that this and other concepts become simpler in case M has dimension

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1, and that all results in this paper are already new in the one-dimensional case.

Our first result asserts that if nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ share CM four "essentially different" finite sets, then there is a nonconstant rational function $g(x) \in \mathbb{C}(x)$ such that $g \circ p = g \circ q$ and $\deg(g)$ is bounded in terms of the sizes of the shared sets; it follows that p and q share CM infinitely many finite sets.

Theorem 1.2. *Pick a positive integer m and nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$. Suppose that p and q share CM each of n finite subsets $S_1, \dots, S_n$ of $\mathbb{C}_\infty$ for some $n \geq 4$, where no S_i is contained in $\cup_{j \neq i} S_j$. Then $g \circ p = g \circ q$ for some nonconstant $g \in \mathbb{C}(x)$ such that $\deg(g) \leq \frac{1}{n-3}(-2 + \sum_{i=1}^n |S_i|)$, where in addition if $n \geq 5$ then $\deg(g) \leq \max_i |S_i|$.*

Note that if $g \circ p = g \circ q$ for some $g \in \mathbb{C}(x) \setminus \mathbb{C}$ then $p^{-1}(g^{-1}(\alpha)) = q^{-1}(g^{-1}(\alpha))$ for each $\alpha \in \mathbb{C}_\infty$, so if α is not a critical value of $g(x)$ then $g^{-1}(\alpha)$ is a set of size $\deg(g)$ which is shared CM by p and q. This yields the following consequence of Theorem 1.2:

Corollary 1.3. *If the conditions of Theorem 1.2 hold then p and q share CM infinitely many pairwise disjoint k-element subsets of $\mathbb{C}_\infty$ for some integer k such that $k \leq \frac{1}{n-2}(-2 + \sum_{i=1}^n |S_i|)$, and if $n \geq 5$ then also $k \leq \max_i |S_i|$.*

Theorem 1.2 is already new when $m = 1$, where it may be viewed as a vast generalization of Nevanlinna's "four values" result and the CM version of his "five values" result. For, if p, q share CM five points then Theorem 1.2 implies that $g \circ p = g \circ q$ with $\deg(g) = 1$, so that $p = q$. Likewise if p, q share CM four points then Theorem 1.2 implies that $g \circ p = g \circ q$ with $\deg(g) \leq 2$. If $\deg(g) = 1$ then we again obtain $p = q$. If $\deg(g) = 2$ then $g = \mu \circ x^2 \circ \nu$ for some Möbius transformations $\mu, \nu \in \mathbb{C}(x)$, so that $x^2 \circ \nu \circ p = x^2 \circ \nu \circ q$ and thus $\nu \circ p = \epsilon\nu \circ q$ for some $\epsilon \in \{1, -1\}$, whence $p = \eta \circ q$ where $\eta := \nu^{-1} \circ \epsilon\nu$ is a Möbius transformation. In a followup paper we will show that our results also imply many other results from the literature, in addition to yielding many new results when one imposes further hypotheses on the sizes of the shared sets S_i. Thus, our results provide a new perspective which connects many old and new results as being consequences of the single general Theorem 1.2.

Theorem 1.2 motivates the following definition:

Definition 1.4. We say that $p, q \in \mathcal{M}(\mathbb{C}^m)$ are *quasi-equivalent* if there exists a nonconstant $g \in \mathbb{C}(x)$ such that $g \circ p = g \circ q$.

We emphasize that quasi-equivalence is much more restrictive than algebraic dependence. For instance, any two rational functions $p, q \in \mathbb{C}(x)$ are algebraically dependent, but the vast majority of such p, q are not quasi-equivalent. Further, as explained before Corollary 1.3, quasi-equivalence is more directly related to value-sharing questions than algebraic dependence. We have seen hundreds of papers about value-sharing which include examples showing that their results would not be true with weaker hypotheses; but we checked that all such examples in these papers consist of quasi-equivalent functions, so it is conceivable that the results of the papers would remain true with weaker hypotheses, once one adds to the conclusion some pairs of quasi-equivalent functions. More generally, it seems natural to seek results showing that certain value-sharing hypotheses imply quasi-equivalence, and conversely to produce examples of non-quasi-equivalent functions with interesting value-sharing properties.

Finally, we note that for applications of Theorem 1.2 it is crucial to have a good bound on $\deg(g)$, in terms of the sizes of the shared sets. It turns out that different types of arguments are needed to prove the existence of $g(x)$ than to bound its degree.

Example 1.5. Theorem 1.2 cannot be improved to three shared sets, since for instance $p(x) := (e^x+2)/(e^x+1)$ does not take the values 1 or 2, so that p and $2p$ share CM $\{\infty\}$, $\{0\}$, and $\{2\}$, but there is no nonconstant $g \in \mathbb{C}(x)$ for which $g \circ p = g \circ 2p$. In this example the meromorphic functions p and $2p$ are algebraically dependent; more generally, we will show in Proposition 3.11 that if nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ are algebraically dependent and share CM three disjoint nonempty finite sets then $g(p) = \alpha g(q)$ for some nonconstant $g \in \mathbb{C}(x)$ and some $\alpha \in \mathbb{C}^*$. A different type of example is $(e^{x^2}-1)/(e^x-1)$ and $(e^{-x^2}-1)/(e^{-x}-1)$, which are algebraically independent but share CM $\{\infty\}$, $\{0\}$, and $\{1\}$.

Many authors have studied pairs of meromorphic functions which share some sets of prescribed sizes. In order to apply our theory to this type of question, and also in order to prove the bounds on $\deg(g)$ in Theorem 1.2, we describe the collection of all sets shared CM by any two quasi-equivalent meromorphic functions p and q. A routine set theory exercise shows that if p and q share two sets S and T, then p and q also share $S \cup T$, $S \cap T$, and $S \setminus T$. Thus every nonempty finite set which is shared CM by p and q can be written as the union of minimal shared sets, where we define a minimal shared set to be a nonempty shared set which does not properly contain any other nonempty shared set. Moreover, distinct minimal shared sets are disjoint, and any union of minimal shared sets is again a shared set. If p and q are quasi-equivalent then let $g(x)$ be a nonconstant rational function of the smallest possible degree such that $g(p) = g(q)$. Let Λ_g be the set of points α in $\mathbb{C}_\infty$ such that g has the same multiplicity at each g-preimage of α; thus, Λ_g includes all points which are not critical values of g (and possibly some critical values as well), so that in particular Λ_g includes all but finitely many points of $\mathbb{C}_\infty$. For each $\alpha \in \mathbb{C}_\infty$, we write $g^{-1}(\alpha)_{\text{set}}$ for the set of distinct g-preimages of α. As explained before Corollary 1.3, the set $g^{-1}(\alpha)_{\text{set}}$ is shared CM by p and q whenever $\alpha \in \Lambda_g$. Conversely, in most situations the collection of such sets $g^{-1}(\alpha)_{\text{set}}$ comprises all minimal shared sets for p and q:

Theorem 1.6. *For quasi-equivalent $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$, let $g(x) \in \mathbb{C}(x)$ be a minimal-degree nonconstant rational function for which $g \circ p = g \circ q$, and define Λ_g as above. Then one of the following occurs:*

(1.6.1) *The collection of all sets $g^{-1}(\alpha)_{set}$ with $\alpha \in \Lambda_g$ equals the collection of all minimal shared sets for p and q.*

(1.6.2) *For some $\beta \in \Lambda_g$, $g^{-1}(\beta)_{set}$ is the union of two distinct minimal shared sets S_1, S_2, and the collection of all minimal shared sets for p and q consists of S_1, S_2, and all sets $g^{-1}(\alpha)_{set}$ with $\alpha \in \Lambda_g \setminus \{\beta\}$.*

In light of Theorems 1.2 and 1.6, in order to describe the possibilities for $p, q, S_1, \ldots, S_4$ where the S_i's are disjoint nonempty finite subsets of $\mathbb{C}_\infty$ which are shared CM by $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$, there are two remaining problems:

(1.7.1) Determine all solutions to $g \circ p = g \circ q$ in nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ and $g \in \mathbb{C}(x) \setminus \mathbb{C}$.

(1.7.2) For each (g, p, q) as in (1.7.1) in which g has minimal degree among all solutions to (1.7.1) for the relevant p and q, determine whether (1.6.2) holds.

There are dozens of papers solving (1.7.1) when g, p, q satisfy additional restrictive properties, for instance [1, 2, 3, 4, 5, 8, 11, 13, 14, 15, 19, 20, 21, 22, 25, 28, 31, 36, 37, 38]. Recent work of the second author and his students goes beyond the cases treated previously, by solving (1.7.1) when any of the following hold:

- the numerator of $(g(x) - g(y))/(x - y)$ is irreducible
- $g(x) = f(x)^n$ for some positive integer n, where there is a primitive n-th root of unity ζ such that $f(p(x)) = \zeta f(q(x))$ and the numerator of $f(x) - \zeta f(y)$ is irreducible
- some $\alpha \in \mathbb{C}_\infty$ has at most two distinct g-preimages.

In the followup paper [35] we determine all situations when (1.6.2) holds in each of the above three cases. An informal conclusion is that (1.6.2) rarely holds, except when β has very few g-preimages. As a consequence, we classify all $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ which share CM each of four disjoint nonempty finite subsets $S_1, \ldots, S_4$ of $\mathbb{C}_\infty$, where in addition at least two S_i's have size 1.

The previous result closest to ours is [17, Thm. 3], which asserts that if $m = 1$ and $n \geq 4$ and some S_i has size 1 then p and q must be algebraically dependent. We note that, since $p := e^x$ and $q := -e^x$ share CM each set $S = \{\alpha, -\alpha\}$ with $\alpha \in \mathbb{C} \setminus \{0\}$, it is not true that if n is big enough and the S_i's have the same size then $p = q$, contradicting the assertion in [18, XXIII].

It would be interesting to seek analogues of our results for shared sets ignoring multiplicities (IM). Some first steps in this direction are taken in [32, 33], but the following questions remain open:

Question 1.8. Is there an absolute constant N so that if nonconstant $p, q \in \mathcal{M}(\mathbb{C})$ share IM N disjoint nonempty finite subsets of $\mathbb{C}_\infty$ then $g \circ p = g \circ q$ for some $g \in \mathbb{C}(x) \setminus \mathbb{C}$?

Question 1.9. If nonconstant $p, q \in \mathcal{M}(\mathbb{C})$ share IM infinitely many finite subsets of $\mathbb{C}_\infty$ then must there be some $g \in \mathbb{C}(x) \setminus \mathbb{C}$ for which $g \circ p = g \circ q$?

Remark 1.10. Question 1.8 is open even in the simplest case when p and q are polynomials. Of course, Question 1.9 is trivially true in that case, since p and q only have finitely many critical values, so that by repeatedly taking intersections and set differences of the given shared sets we obtain infinitely many IM-shared sets which contain no critical values and hence are shared CM, whence the conclusion follows from our results (or in this case from the easier Lemma 3.9). Finally, the multivariable analogues of these questions are also open, but we stated the one-variable cases to focus attention on the fundamental difficulties.

This paper is organized as follows. In the next section we list the notation and terminology we will use. In Section 3 we show that if $p, q \in \mathcal{M}(\mathbb{C}^m)$ share four disjoint finite sets then $g \circ p = g \circ q$ for some nonconstant $g \in \mathbb{C}(x)$. Our proof combines several new ideas with ingredients from Nevanlinna's proof of his "four values" theorem, which in turn was based an earlier argument due to Pólya [29]. Our proof yields no bound on $\deg(g)$ in terms of the sizes of the shared sets, and the next four sections are required to prove such a bound. In Section 4 we describe the collection of all rational functions $g(x)$ which satisfy $g \circ p = g \circ q$ for prescribed meromorphic functions p, q on an arbitrary complex manifold $\mathcal{R}$. In Section 5 we prove some useful properties about multiplicities of preimages of points under a minimal-degree nonconstant $g(x) \in \mathbb{C}(x)$ satisfying $g \circ p = g \circ q$. The results in Section 4 and especially Section 5 are of independent interest; certainly the combination of Galois-theoretic and topological methods used in these sections is quite different from previous work in the subject. In Section 6 we describe the collection of all sets shared CM by any prescribed $p, q \in \mathcal{M}(\mathbb{C}^m)$ for which $g \circ p = g \circ q$ for some $g \in \mathbb{C}(x) \setminus \mathbb{C}$, and prove a refinement of Theorem 1.6. Finally, in Section 7 we combine the results of the previous sections in order to prove a generalization of Theorem 1.2.

2. Notation and terminology

In this section we list the notation and terminology used in this paper. These are also defined when first used, but we list them here for ease of reference.

We first recall the standard definition of multiplicity of points under a meromorphic function.

Definition 2.1. Let M be a complex manifold, let $p \colon M \to \mathbb{C}$ be a holomorphic function which is not identically zero, and let α be a point in $\mathcal{Z}_p := \{\beta \in M : p(\beta) = 0\}$. Further, let $\mathcal{O}_\alpha$ be the local ring consisting of the germs at α of holomorphic functions defined on a neighborhood of α, and let $\mathcal{I}$ be the ideal of $\mathcal{O}_\alpha$ consisting of all elements which vanish on $\mathcal{Z}_p$. Letting k be the maximal integer for which $p \in \mathcal{I}^k$, we say that p has a zero of multiplicity k at α, and write $m_p(\alpha) := k$. If $\alpha \in M \setminus \mathcal{Z}_p$ then we define $m_p(\alpha) := 0$.

For any meromorphic function $p \in \mathcal{M}(M)$ and a point $\alpha \in M$ for which $p(\alpha) \in \mathbb{C}$, we may write $p - p(\alpha)$ in a neighborhood of α as the quotient of two holomorphic functions q/r, and the multiplicity of p at α is $\nu_p(\alpha) := m_q(\alpha) - m_r(\alpha)$. Finally, if $p(\alpha) = \infty$ then the multiplicity of p at α is $\nu_p(\alpha) := \nu_{1/p}(\alpha)$.

- $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$
- $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ is the Riemann sphere
- all complex manifolds in this paper are assumed to be connected. M denotes a general complex manifold and $\mathcal{R}$ denotes a Riemann surface
- $\mathcal{M}(M)$ is the set of all meromorphic functions on the complex manifold M
- for $p \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ we write $\mathcal{E}(p) := \mathbb{C}_\infty \setminus p(\mathbb{C}^m)$ for what is sometimes called the set of Picard exceptional values of p; Picard's little theorem says $|\mathcal{E}(p)| \le 2$
- a *multiset* (or "set with multiplicities") is a collection of elements which need not be distinct
- $p^{-1}(\alpha)$ is the multiset of all preimages of $\alpha \in \mathbb{C}_\infty$ under some nonconstant $p \in \mathcal{M}(M)$, counted with multiplicities
- S_{set} is the set of distinct elements in the multiset S
- if S is a nonempty finite multiset then $\operatorname{gcdmult}(S)$ denotes the greatest common divisor of the multiplicities of all elements of S
- if S is a multiset and k is a positive integer then S^k denotes the union of k copies of S
- $\{a^{*m}, b\}$ is the multiset having m copies of a and one copy of b
- $\mathcal{G}_1(p, q)$ is defined in Definition 4.1
- a *coequalizer* of p and q is a rational function $g \in \mathcal{G}_1(p, q)$ of minimal degree (see Remark 4.3)
- minimal shared multisets are defined in Definition 6.1
- the multisets T_α are defined in Definition 6.3
- $T_n(x)$ is the degree-n Chebyshev polynomial, namely the unique polynomial such that $T_n(\cos\theta) = \cos n\theta$.

3. Four shared sets implies infinitely many

In this section we prove that if nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ share four disjoint finite sets then they share infinitely many, by showing that there must be a nonconstant $g \in \mathbb{C}(x)$ for which $g(p) = g(q)$. In fact we prove a generalization of this assertion, in which the sets S_i are replaced by *multisets*, i.e., collections of elements that need not be distinct. The proof in this section does not bound the degree of g, but in later sections we will use the existence of g in order to deduce such a bound. The main result of this section is as follows.

Theorem 3.1. *If nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ share CM each of four finite multisets $S_1, \dots, S_4$ of elements of $\mathbb{C}_\infty$, where no S_i is contained in the union of the other S_j's, then $g \circ p = g \circ q$ for some nonconstant $g \in \mathbb{C}(x)$. Conversely, if $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ and $g \in \mathbb{C}(x) \setminus \mathbb{C}$ satisfy $g \circ p = g \circ q$ then p and q share CM each of infinitely many pairwise disjoint k-element subsets of $\mathbb{C}_\infty$, where $k := \deg(g)$.*

The proof of Theorem 3.1 relies on the following several-variable generalization (see [12, Thm. 3.5] or [16, p. 54]) of a classical result of Borel [7]:

Lemma 3.2. *For any $n > 0$, if $r_1, \dots, r_n$ are entire functions on $\mathbb{C}^m$ which have no zeroes, and $r_1 + \dots + r_n = 0$, then $r_i = \alpha r_j$ for some $i \neq j$ and some $\alpha \in \mathbb{C}^*$.*

We begin by adapting this result to our setting. It is convenient to use the language of divisors.

Definition 3.3. For any complex manifold M, the *divisor* of a nonconstant $p \in \mathcal{M}(M)$ is the (possibly infinite) formal $\mathbb{Z}$-linear combination of points of M defined as the sum of the zeroes of p minus the sum of the poles of p, where the zeroes and poles are counted with multiplicities. If p is introduced as an element of $\mathbb{C}(x)$ then we view p as an element of $\mathcal{M}(\mathbb{C}_\infty)$ when defining its divisor – thus, in this situation we allow ∞ as a possible zero or pole of p, although we would not allow this if the same function p were instead introduced as an element of $\mathcal{M}(\mathbb{C})$.

Lemma 3.4. *Pick $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ and $f_i, g_i \in \mathbb{C}(x) \setminus \mathbb{C}$ (for $i = 1, 2, 3$), and suppose that for each i the divisor of $f_i(p)$ equals the divisor of $g_i(q)$. Then there exist integers n_1, n_2, n_3 which are not all zero and for which $F(p)/G(q)$ is in $\mathbb{C}^*$, where $F := \prod_{i=1}^3 f_i^{n_i}$ and $G := \prod_{i=1}^3 g_i^{n_i}$. If in addition each f_i has at least one zero or pole which is not a zero or pole of any other f_j, then F and G are nonconstant.*

Proof. Write $h_i(x, y) := f_i(x)/g_i(y)$ for $i = 1, 2, 3$. Since the field extension $\mathbb{C}(x, y)/\mathbb{C}$ has transcendence degree 2, the three elements $h_i \in \mathbb{C}(x, y)$ must be algebraically dependent. Thus there is a nonzero polynomial $P(u, v, w) \in \mathbb{C}[u, v, w]$ such that $P(h_1, h_2, h_3) = 0$. Writing $P(u, v, w) := \sum c_{r,s,t} u^r v^s w^t$ where the sum is over a nonempty finite set Δ of triples (r, s, t) of nonnegative integers, and each $c_{r,s,t}$ is in $\mathbb{C}^*$, it follows that $\sum c_{r,s,t} h_1^r h_2^s h_3^t = 0$. Recall that the h_i's are in $\mathbb{C}(x, y)$, and substitute p for x and q for y to obtain $\sum c_{r,s,t} H_1^r H_2^s H_3^t = 0$ where $H_i := h_i(p, q) = f_i(p)/g_i(q)$. Since $f_i(p)$ and $g_i(q)$ have the same divisor, their ratio H_i has no zeroes or poles. Thus, for each triple $(r, s, t) \in \Delta$, the function $c_{r,s,t} H_1^r H_2^s H_3^t$ is entire and has no zeroes, so by Lemma 3.2 there are two distinct triples (r, s, t) and (r', s', t')

in Δ for which $H_1^r H_2^s H_3^t = \alpha H_1^{r'} H_2^{s'} H_3^{t'}$ with $\alpha \in \mathbb{C}^*$. Writing $n_1 := r - r'$, $n_2 := s - s'$, and $n_3 := t - t'$, it follows that $\prod_{i=1}^3 H_i^{n_i} = \alpha$, or equivalently $F(p)/G(q) = \alpha$ where $F := \prod_{i=1}^3 f_i^{n_i}$ and $G := \prod_{i=1}^3 g_i^{n_i}$. Here n_1, n_2, n_3 are integers which are not all zero.

Now suppose that each f_i has at least one zero or pole δ_i which is not a zero or pole of any other f_j. Since at least one n_i is nonzero, it follows that the corresponding δ_i is a zero or pole of F, so that F is nonconstant. $G(q) = F(p)/\alpha$ is also nonconstant, so that G is nonconstant as well. □

In order to apply Lemma 3.4 to specific $p, q \in \mathcal{M}(M)$, we need to exhibit $f_i, g_i \in \mathbb{C}(x)$ for which $f_i(p)$ and $g_i(q)$ have the same divisor. In our situation, f_i will be a product of integer powers of the characteristic polynomials of some shared multisets. By a slight abuse of notation, if S is a finite multiset of elements of a complex manifold M then we also write S for the divisor on M defined as the formal sum of the elements of the multiset S.

Lemma 3.5. *Let p and q be nonconstant meromorphic functions on a complex manifold M, and let S_1 and S_2 be disjoint nonempty finite multisets of elements of $\mathbb{C}_\infty$ such that p and q share each S_i CM. Then there are integers $n_1, n_2 > 0$ and a nonconstant $h \in \mathbb{C}(x)$ such that the divisor of $h(x)$ is $n_1 S_1 - n_2 S_2$ and the divisors of $h(p)$ and $h(q)$ are equal. In particular, when M is compact, there exists a $\gamma \in \mathbb{C}^*$ for which $h(p) = \gamma h(q)$.*

Proof. First assume that neither S_i contains ∞. Let $f_i(x) := \prod_{\alpha \in S_i}(x - \alpha)$ be the characteristic polynomial of S_i. By hypothesis, the f_i's are nonconstant coprime polynomials such that, for each i, $f_i \circ p$ and $f_i \circ q$ have the same zeroes CM. Then $h(x) := f_1(x)^{\deg f_2}/f_2(x)^{\deg f_1}$ is a nonconstant rational function whose numerator and denominator are monic polynomials of the same degree, so that $h(\infty) = 1$. Thus the zeroes of $h(p)$ coincide CM with the zeroes of $f_1(p)^{\deg f_2}$, which coincide CM with the zeroes of $f_1(q)^{\deg f_2}$, and hence with the zeroes of $h(q)$. Likewise, the poles of $h(p)$ agree CM with the poles of $h(q)$. Since the zeroes of h consist of $|S_2|$ copies of S_1, and the poles of h consist of $|S_1|$ copies of S_2, this proves the result in case neither S_i contains ∞.

If some S_i contains ∞ then let $\mu(x)$ be a Möbius transformation such that $\mu(S_1 \cup S_2)$ does not contain ∞. Then $\widehat{p} := \mu \circ p$ and $\widehat{q} := \mu \circ q$ share CM each multiset $\widehat{S_i} := \mu(S_i)$, where the $\widehat{S_i}$'s are nonempty and disjoint but do not contain ∞. Thus there is a nonconstant $\widehat{h} \in \mathbb{C}(x)$ such that $\widehat{h}(\widehat{p})$ and $\widehat{h}(\widehat{q})$ have the same divisor, where in addition the divisor of $\widehat{h}$ is $n_1\widehat{S_1} - n_2\widehat{S_2}$ for some positive integers n_1, n_2. Then $h := \widehat{h} \circ \mu$ has divisor $n_1 S_1 - n_2 S_2$, and the divisors of $h(p)$ and $h(q)$ are identical.

Finally, when M is compact, $\gamma := h(p)/h(q)$ is a holomorphic map $M \to \mathbb{C}_\infty$ which has no zeroes or poles, so the compactness of M implies $\gamma \in \mathbb{C}^*$. □

In fact, as can be seen from the proof, n_1 and n_2 can be chosen as $|S_2|$ and $|S_1|$, respectively. With these ingredients in hand, we now prove that if p and q share four multisets then we obtain a weaker version of our desired functional equation.

Proposition 3.6. *For nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$, and any pairwise disjoint nonempty finite multisets $S_1, \dots, S_4$ of elements of $\mathbb{C}_\infty$ such that p, q share CM each S_i, there exist $h \in \mathbb{C}(x) \setminus \mathbb{C}$ and $\gamma \in \mathbb{C}^*$ such that $h \circ p = \gamma h \circ q$. Moreover, h can be chosen so that its divisor is a $\mathbb{Z}$-linear combination of S_1, S_2, S_3, S_4.*

Proof. By Lemma 3.5, for each $i = 1, 2, 3$ there exist a nonconstant $h_i \in \mathbb{C}(x)$ and positive integers u_i, v_i such that $h_i(x)$ has divisor $u_i S_i - v_i S_4$ and the divisors of $h_i(p)$ and $h_i(q)$ equal one another. By Lemma 3.4, there are integers n_1, n_2, n_3 which are not all zero and for which $h := \prod_{i=1}^3 h_i^{n_i}$ is nonconstant and $h(p) = \gamma \cdot h(q)$ for some $\gamma \in \mathbb{C}^*$. Since the divisor of h is $\sum_{i=1}^3 (n_i u_i S_i - n_i v_i S_4)$, this yields the result. $\square$

Our proof of Theorem 3.1 also uses the following result of Coman and Poletsky [9, Thm. 5.2]:

Lemma 3.7. *If nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ are algebraically dependent then there exist a compact Riemann surface $\mathcal{R}$ of genus 0 or 1, a holomorphic map $r \colon \mathbb{C}^m \to \mathcal{R}$, and $p_0, q_0 \in \mathcal{M}(\mathcal{R})$ such that $p = p_0 \circ r$ and $q = q_0 \circ r$.*

Remark 3.8. The special case $m = 1$ of Lemma 3.7 was proved in [6, Thm. 1] independently and simultaneously to [9].

In order to apply Lemma 3.7 to questions about shared multisets, we first address shared multisets on a compact complex manifold.

Lemma 3.9. *Let M be a compact complex manifold, and pick $p_0, q_0 \in \mathcal{M}(M) \setminus \mathbb{C}$. If S_1, S_2, S_3 are disjoint nonempty finite multisets of elements of $\mathbb{C}_\infty$ such that p_0, q_0 share CM S_1 and S_2, and $p_0^{-1}(S_3)_{set} \subseteq q_0^{-1}(S_3)_{set}$, then $g \circ p_0 = g \circ q_0$ for some nonconstant $g \in \mathbb{C}(x)$.*

Proof. By Lemma 3.5 there exists a nonconstant rational function h and a $\gamma \in \mathbb{C}^*$ such that $h \circ p_0 = \gamma h \circ q_0$. Since M is compact, each element of S_3 has the form $s = p_0(\theta)$ with $\theta \in M$, so that $h(p_0(\theta)) = \gamma h(q_0(\theta)) \in \gamma h(S_3)$. Thus $h(S_3)_{\text{set}} \subseteq \gamma h(S_3)_{\text{set}}$. Since by Lemma 3.5 all zeroes and poles of h are in $S_1 \cup S_2$, the set $h(S_3)_{\text{set}}$ is contained in $\mathbb{C}^*$. Since this set is finite and nonempty, and is preserved by multiplication by γ, it follows that $\gamma^n = 1$ for some positive integer n, so that $h^n \circ p_0 = h^n \circ q_0$. $\square$

We also use the following generalization of Picard's little theorem:

Lemma 3.10. *If $\mathcal{R}$ is a compact Riemann surface and $h \colon \mathbb{C}^m \to \mathcal{R}$ is a nonconstant holomorphic map which is not surjective, then there exists a biholomorphic map $\mathcal{R} \to \mathbb{C}_\infty$, and $\mathcal{R} \setminus h(\mathbb{C}^m)$ has size at most 2.*

Proof. For any nonempty finite subset S of $\mathcal{R} \setminus h(\mathbb{C}^m)$, write $\mathcal{R}_0 := \mathcal{R} \setminus S$. Then h induces a nonconstant holomorphic map $\mathbb{C}^m \to \mathcal{R}_0$, so $\mathcal{R}_0$ cannot be hyperbolic (e.g. by [26, Lemma 2.3]). Thus $\mathcal{R}$ has genus zero (so $\mathcal{R} \cong \mathbb{C}_\infty$) and S has size at most 2. □

Proof of Theorem 3.1. If $g \circ p = g \circ q$ then p and q share CM the multiset $S_\alpha := g^{-1}(\alpha)$ for any $\alpha \in \mathbb{C}_\infty$. Plainly $|S_\alpha| = \deg(g)$ and $S_\alpha \cap S_\beta = \emptyset$ when $\alpha \neq \beta$, and moreover S_α is a set whenever α is not one of the finitely many critical values of g. Thus p and q share CM infinitely many pairwise disjoint sets, each of which has size $\deg(g)$.

Conversely, we now assume that p and q share CM each of four pairwise disjoint finite multisets $S_1, \dots, S_4$ of elements of $\mathbb{C}_\infty$, where in addition no S_i is contained in the union of the other S_j's. Proposition 3.6 yields $h \in \mathbb{C}(x) \setminus \mathbb{C}$ and $\gamma \in \mathbb{C}^*$ such that $h \circ p = \gamma h \circ q$, and thus p and q are algebraically dependent. By Lemma 3.7, there exist a compact Riemann surface $\mathcal{R}$, a holomorphic map $r\colon \mathbb{C}^m \to \mathcal{R}$, and $p_0, q_0 \in \mathcal{M}(\mathcal{R})$ such that $p = p_0 \circ r$ and $q = q_0 \circ r$. Since p and q are nonconstant, also p_0, q_0, r are nonconstant. The identity $h \circ p = \gamma h \circ q$ now becomes $h \circ p_0 \circ r = \gamma h \circ q_0 \circ r$, so that $h \circ p_0 = \gamma h \circ q_0$. Since $\mathcal{R}$ is compact, we can speak of the degrees of p_0 and q_0 (i.e., the numbers of preimages of any point, counted with multiplicities), and the above identity implies $\deg(h) \cdot \deg(p_0) = \deg(h) \cdot \deg(q_0)$, whence $\deg(p_0) = \deg(q_0)$.

For any finite multiset S of elements of $\mathbb{C}_\infty$, the multiset $p^{-1}(S)$ is the union of all $r^{-1}(\alpha)$ with $\alpha \in p_0^{-1}(S)$. Thus S is shared CM by p and q if and only if the multiset differences $p_0^{-1}(S) \setminus q_0^{-1}(S)$ and $q_0^{-1}(S) \setminus p_0^{-1}(S)$ each consist of elements of $\mathcal{E} := \mathcal{R} \setminus r(\mathbb{C}^m)$. Since $p_0^{-1}(S)$ and $q_0^{-1}(S)$ have the same size, and they also have the same size after removing all copies of elements of $\mathcal{E}$ from both of them, it follows that $p_0^{-1}(S)$ and $q_0^{-1}(S)$ contain the same number of elements of $\mathcal{E}$ (when counted with multiplicities).

We may assume that at most two of the S_i's are shared CM by p_0 and q_0, since otherwise Lemma 3.9 produces $g \in \mathbb{C}(x) \setminus \mathbb{C}$ with $g \circ p_0 = g \circ q_0$, whence also $g \circ p = g \circ q$. By relabeling the S_i's if needed, we may assume that for $i \in \{1, 2\}$ we have $p_0^{-1}(S_i) \neq q_0^{-1}(S_i)$, so that $p_0^{-1}(S_i) \setminus q_0^{-1}(S_i)$ and $q_0^{-1}(S_i) \setminus p_0^{-1}(S_i)$ are disjoint nonempty multisets of the same size which each consist of elements of $\mathcal{E}$. We have $|\mathcal{E}| \leq 2$ by Lemma 3.10, and also the four multisets $p_0^{-1}(S_i)$ are pairwise disjoint, as are the four multisets $q_0^{-1}(S_i)$. Thus there are distinct $\alpha_1, \alpha_2 \in \mathcal{E}$, and positive integers e_1, e_2, such that for each $i \in \{1, 2\}$

- $p_0^{-1}(S_i) \setminus q_0^{-1}(S_i)$ consists of e_i copies of α_i, and
- $q_0^{-1}(S_i) \setminus p_0^{-1}(S_i)$ consists of e_i copies of α_{3-i}.

Since $\mathcal{E} = \{\alpha_1, \alpha_2\}$ is contained in $p_0^{-1}(S_1 \cup S_2)$ and $q_0^{-1}(S_1 \cup S_2)$, it follows that $p_0^{-1}(S_j) = q_0^{-1}(S_j)$ for $j \in \{3, 4\}$. Next, for $T := S_1 \cup S_2$, the multiset $p_0^{-1}(T)$ is the union of $\cup_{i=1}^2 (p_0^{-1}(S_i) \cap q_0^{-1}(S_i))$ with $e_1 + e_2$ copies of each α_i, and this union also equals $q_0^{-1}(T)$. Hence p_0 and q_0 share CM the disjoint multisets T, S_3, and S_4, so by Lemma 3.9 there exists $g \in \mathbb{C}(x) \setminus \mathbb{C}$ such that $g \circ p_0 = g \circ q_0$, whence also $g \circ p = g \circ q$. $\square$

We conclude this section with a variant of Theorem 3.1 addressing algebraically dependent meromorphic functions which share three multisets. This result will not be used elsewhere in this paper.

Proposition 3.11. *Suppose algebraically dependent $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ share CM three disjoint nonempty finite multisets S_1, S_2, S_3 of elements of $\mathbb{C}_\infty$. Then $g(p) = \alpha g(q)$ for some nonconstant $g(x) \in \mathbb{C}(x)$ and some $\alpha \in \mathbb{C}^*$.*

Proof. By Lemma 3.7, we can write $p = p_0 \circ r$ and $q = q_0 \circ r$ for some compact Riemann surface $\mathcal{R}$, some holomorphic map $r\colon \mathbb{C}^m \to \mathcal{R}$, and some $p_0, q_0 \in \mathcal{M}(\mathcal{R})$. Writing $\mathcal{E} := \mathcal{R} \setminus r(\mathbb{C}^m)$, put $A_i := \mathcal{E} \cap p_0^{-1}(S_i)$ and $B_i := \mathcal{E} \cap q_0^{-1}(S_i)$. For each $i \in \{1, 2, 3\}$, one of the following holds:

(1) $A_i = B_i = \emptyset$
(2) $A_i \neq \emptyset = B_i$
(3) $A_i = \emptyset \neq B_i$
(4) $A_i \neq \emptyset$ and $B_i \neq \emptyset$.

Since the multisets $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ agree except for copies of elements of A_i in $p_0^{-1}(S_i)$ and elements of B_i in $q_0^{-1}(S_i)$, we see that

- if (1) holds then $p_0^{-1}(S_i) = q_0^{-1}(S_i)$
- if (2) holds then $|p_0^{-1}(S_i)| > |q_0^{-1}(S_i)|$
- if (3) holds then $|p_0^{-1}(S_i)| < |q_0^{-1}(S_i)|$.

Since $|p_0^{-1}(S_i)| = \deg(p_0) \cdot |S_i|$, it follows that

- if (1) holds then $\deg(p_0) = \deg(q_0)$
- if (2) holds then $\deg(p_0) > \deg(q_0)$
- if (3) holds then $\deg(p_0) < \deg(q_0)$.

Thus there cannot be i, j for which two different cases among (1),(2),(3) hold. Since $|\mathcal{E}| \leq 2$ by Lemma 3.10, there is at least one i for which A_i is empty, so that (1) or (3) holds for that i; and likewise there is at least one j for which B_j is empty, so that (1) or (2) holds for that j. Thus (1) holds for at least one i, and every j satisfies either (1) or (4). Write $f_i(x) := \prod_{\alpha \in S_i} (x - \alpha)$, and put $n_i := |S_i|$. If $p_0^{-1}(S_i) = q_0^{-1}(S_i)$ for at least two i's, say $i = 1$ and $i = 2$, then for $g := (f_2)^{n_1}/(f_1)^{n_2}$ we see that $g \circ p_0$ and $g \circ q_0$ have the same divisor, so their ratio is constant by compactness of $\mathcal{R}$, yielding the desired conclusion. Henceforth assume that there is exactly one i for which $p_0^{-1}(S_i) = q_0^{-1}(S_i)$. We may assume that (1) holds for $i = 3$ but (4) holds for $i = 1$ and $i = 2$. Then A_1, B_1, A_2, B_2 each have size 1, and $A_1 \cup A_2$ and $B_1 \cup B_2$ are the same two-element set. Here $\deg(p_0) = \deg(q_0)$, so that $|p_0^{-1}(S_i)| = |q_0^{-1}(S_i)|$ for each i, whence since the multisets of elements of $\mathbb{C}_\infty \setminus \mathcal{E}$ in $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ coincide, it follows that the multisets of elements of $\mathcal{E}$ in $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ have the same size. Since $A_1 \cap B_1 = \emptyset$, we have $A_1 = B_2 = \{\alpha_1\}$ and $A_2 = B_1 = \{\alpha_2\}$, where, for $i \in \{1, 2\}$ and some positive integer e_i, the multisets $p_0^{-1}(S_i) \setminus q_0^{-1}(S_i)$ and

$q_0^{-1}(S_i)\setminus p_0^{-1}(S_i)$ consist of e_i copies of α_i and e_i copies of α_{3-i}, respectively. Putting $h := (f_1)^{e_2}(f_2)^{e_1}$, it follows that $h(p_0)$ and $h(q_0)$ have the same zeroes CM, so for $g := h^{n_3}/(f_3)^{\deg(h)}$ the functions $g(p_0)$ and $g(q_0)$ have the same divisor and hence have constant ratio. Finally, $g(x)$ is nonconstant since each element of S_3 is a pole of $g(x)$. □

4. Minimal relations between meromorphic functions

Theorem 3.1 yields nonconstant rational functions $g(x)$ such that $g(p) = g(q)$, for prescribed $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ satisfying certain shared-multiset hypotheses. In this section we describe the collection of all rational functions $g(x)$ satisfying $g(p) = g(q)$.

Definition 4.1. For any complex manifold M and any nonconstant $p, q \in \mathcal{M}(M)$, let $\mathcal{G}_1(p, q)$ be the set of all $g \in \mathbb{C}(x) \setminus \mathbb{C}$ such that $g \circ p = g \circ q$. When the choices of p and q are clear, we write $\mathcal{G}_1$ for $\mathcal{G}_1(p, q)$.

Proposition 4.2. *Let M be a complex manifold, and pick $p, q \in \mathcal{M}(M)\setminus\mathbb{C}$. If $\mathcal{G}_1$ is nonempty and $g_1(x)$ is a minimal-degree element of $\mathcal{G}_1$ then $\mathcal{G}_1 = \{d \circ g_1 : d \in \mathbb{C}(x) \setminus \mathbb{C}\}$.*

Proof. Let $L = \mathcal{G}_1 \cup \mathbb{C}$ be the set of all $g(x) \in \mathbb{C}(x)$ for which $g \circ p = g \circ q$. Then L contains $\mathbb{C}$ and is preserved by addition, multiplication, and division by nonzero elements, so L is a field between $\mathbb{C}$ and $\mathbb{C}(x)$. Since $L \neq \mathbb{C}$ by hypothesis, Lüroth's theorem [30, Thm. 2] implies $L = \mathbb{C}(h(x))$ for some nonconstant $h(x) \in L$. For any minimal-degree $g_1 \in \mathcal{G}_1$, since $g_1 \in L$ we have $g_1 = \mu \circ h$ for some nonconstant $\mu \in \mathbb{C}(x)$. The minimality of $\deg(g_1)$ implies $\mu(x)$ is a Möbius transformation, so that $L = \mathbb{C}(g_1(x))$, which implies the conclusion. □

Remark 4.3. This proposition shows, when M is a Riemann surface, that g_1 is a coequalizer of $p, q : \mathbb{C} \to \mathbb{C}_\infty$ in the category **Riem** of Riemann surfaces and holomorphisms between them, since maps from $\mathbb{C}_\infty$ to a Riemann surface $\mathcal{R}$ is trivial unless $\mathcal{R} \cong \mathbb{C}_\infty$. Although when $\dim M > 1$, the g_1 will no longer be a category-theoretical coequalizer, by a slight abuse of terminology we will still refer to it as a coequalizer.

5. Complete multiple values of the minimal-degree rational function relating p and q

In this section we prove a result about the multiplicities of points under a minimal-degree $g \in \mathcal{G}_1$; this will be used in our proof of Theorem 1.2. Recall that if S is a multiset then S_{set} denotes the underlying set, and $\operatorname{gcdmult}(S)$ denotes the greatest common divisor of the multiplicities of all the elements of S.

Proposition 5.1. *For a complex manifold M, quasi-equivalent nonconstant $p, q \in \mathcal{M}(M)$, and their coequalizer $g \in \mathbb{C}(x)$, there are at most two points $\alpha \in \mathbb{C}_\infty$ for which $\operatorname{gcdmult}(g^{-1}(\alpha)) > 1$.*

We will deduce Proposition 5.1 from the following result, which is of independent interest.

Proposition 5.2. *Pick a nonconstant $g \in \mathbb{C}(x)$ and distinct $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}_\infty$. Suppose that $e_i := \operatorname{gcdmult}(g^{-1}(\alpha_i))$ is at least 2 for each $i = 1, 2, 3$. Then the triple (e_1, e_2, e_3) is a permutation of an element of*

$$\mathcal{N} := \{(2, 2, r) \colon r > 1\} \cup \{(2, 3, s) \colon 3 \le s \le 5\}.$$

Let π be a permutation of $\{1, 2, 3\}$ such that the triple $N := (e_{\pi(1)}, e_{\pi(2)}, e_{\pi(3)})$ is in $\mathcal{N}$, and let $\mu(x)$ be the unique Möbius transformation which maps the points $\alpha_{\pi(1)}$, $\alpha_{\pi(2)}$, $\alpha_{\pi(3)}$ to 1, 0, ∞, respectively. Then $\mu \circ g = f_N \circ h$ for some $h \in \mathbb{C}(x)$, where

$$\begin{aligned} f_{(2,2,r)} &= \frac{(x^r + 1)^2}{4x^r} \\ f_{(2,3,3)} &= \frac{(x^4 + 8x)^3}{64(x^3 - 1)^3} \\ f_{(2,3,4)} &= \frac{(x^8 + 14x^4 + 1)^3}{108(x^5 - x)^4} \\ f_{(2,3,5)} &= \frac{(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)^3}{-1728(x^{11} + 11x^6 - x)^5}. \end{aligned}$$

Conversely, for each $N \in \mathcal{N}$ we have

$$\begin{aligned} \operatorname{gcdmult}(f_N^{-1}(1)) &= N(1) \\ \operatorname{gcdmult}(f_N^{-1}(0)) &= N(2) \\ \operatorname{gcdmult}(f_N^{-1}(\infty)) &= N(3), \end{aligned}$$

and there is a finite set T_N of Möbius transformations such that

$$f_N(x) - f_N(y) = \frac{\prod_{\nu \in T_N} (x - \nu(y))}{D_N(x)},$$

where $D_N(x)$ is the denominator exhibited in the definition of $f_N(x)$. Finally, for each $\nu \in T_N$ there is a positive integer k with $k < \deg(f_N)$ such that the composition $\nu \circ \nu \circ \cdots \circ \nu$ of k copies of ν equals x.

Remark 5.3. The rational functions $f_N(x)$ in Proposition 5.2 date back at least to the 19-th century book of Klein [23]. These rational functions generate the fields of rational functions invariant under the non-cyclic finite rotation groups of the sphere, namely the groups of rotational symmetries of the regular dihedron, tetrahedron, octahedron, or icosahedron. Thus the field extension $\mathbb{C}(x)/\mathbb{C}(f_N(x))$ is Galois with Galois group D_r, A_4, S_4 or A_5 according as N is $(2, 2, r)$, $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$; moreover, the elements of the Galois group are the maps $x \mapsto \nu(x)$ with $\nu \in T_N$. For a beautiful exposition of this material, see [34].

Proof that Proposition 5.2 implies Proposition 5.1. Let $\alpha_1, \alpha_2, \alpha_3$ be distinct points in $\mathbb{C}_\infty$, and suppose that each value $e_i := \operatorname{gcdmult}(g^{-1}(\alpha_i))$ is greater than 1. Proposition 5.2 implies that $\mu \circ g = f_N \circ h$ for some Möbius transformation $\mu(x)$, some $N \in \mathcal{N}$, and some $h \in \mathbb{C}(x)$. Since $g(p) = g(q)$, we have $f_N(h(p)) = f_N(h(q))$, so that $\prod_{\nu \in T_N}(h(p) - \nu(h(q))) = 0$, and thus $h(p) = \nu(h(q))$ for some $\nu \in T_N$. By Proposition 5.2, the order of $\nu(x)$ under composition is an integer k which is less than $\deg(f_N)$. We will give two different proofs that this information yields a contradiction, one using Galois theory and one from first principles.

We first give the algebraic proof. The function $\sigma\colon \mathbb{C}(x) \to \mathbb{C}(x)$ defined by $\sigma(u(x)) := u(\nu(x))$ is an order-k automorphism of the field $\mathbb{C}(x)$. Writing L for the set of elements of $\mathbb{C}(x)$ fixed by σ, Artin's theorem from Galois theory [24, Thm. VI.1.8] implies that L is a subfield of $\mathbb{C}(x)$ such that $[\mathbb{C}(x) : L] = k$. Since L properly contains $\mathbb{C}$, by Lüroth's theorem we have $L = \mathbb{C}(u(x))$ for some nonconstant $u(x) \in \mathbb{C}(x)$, and it is known that $[\mathbb{C}(x) : \mathbb{C}(u(x))] = \deg(u)$. But then $u(h(p)) = u(\nu(h(q)) = \sigma(u)(h(q)) = u(h(q))$, which contradicts minimality of $\deg(g)$ since $\deg(u \circ h) = k \cdot \deg(h) < \deg(f_N) \cdot \deg(h) = \deg(g)$.

We now give the self-contained proof. If $\nu(\infty) \neq \infty$ then the numerator of the rational function $\nu(x) - x$ has degree 2 and hence has a zero in $\mathbb{C}$. Thus in any case the set S of fixed points of $\nu(x)$ is nonempty. Let $\rho(x)$ be a Möbius transformation such that $\rho(\infty) \in S$ and if $|S| > 1$ then also $\rho(0) \in S$. Then $\theta := \rho^{-1} \circ \nu \circ \rho$ is a Möbius transformation having $|S|$ fixed points and having the same order under composition as does $\nu(x)$, which by Proposition 5.2 is an integer k less than $\deg(f_N)$. If $|S| = 1$ then ∞ is the unique fixed point of $\theta(x)$, so that $\theta(x)$ is a degree-one polynomial and $\theta(x) - x$ is a nonzero constant β, whence $\theta(x) = x + \beta$ has infinite order under composition, contradiction. Thus $|S| > 1$, so $\theta(x)$ fixes 0 and ∞, and hence $\theta(x) = \zeta x$ for some $\zeta \in \mathbb{C}^*$. Plainly the order of $\theta(x)$ under composition is the order of ζ under multiplication, so that ζ is a primitive k-th root of unity. Since $\rho^{-1}(h(p)) = \zeta\rho^{-1}(h(q))$, it follows that $\mathcal{G}_1$ contains $x^k \circ \rho^{-1} \circ h$, contradicting minimality of $\deg(g)$. □

We have now reduced the proof of Proposition 5.1 to the proof of Proposition 5.2. Our proof of the latter result uses the following version of the Hurwitz genus formula for holomorphic maps $\mathbb{C}_\infty \to \mathbb{C}_\infty$:

Lemma 5.4. *Any $g \in \mathbb{C}(x)$ of degree $k > 0$ satisfies*

$$2k - 2 = \sum_{\alpha \in \mathbb{C}_\infty} \bigl(k - |g^{-1}(\alpha)_{set}|\bigr).$$

The Hurwitz formula immediately implies that for any nonconstant rational function $g(x)$, there cannot be four distinct points $\alpha \in \mathbb{C}_\infty$ for which $|g^{-1}(\alpha)_{\text{set}}| \le \deg(g)/2$, and hence there cannot be four distinct $\alpha \in \mathbb{C}_\infty$ for which $\operatorname{gcdmult}(g^{-1}(\alpha)) > 1$. However, there do exist nonconstant $g \in \mathbb{C}(x)$ for which $\operatorname{gcdmult}(g^{-1}(\alpha)) > 1$ for three distinct $\alpha \in \mathbb{C}_\infty$, and the goal of

Proposition 5.2 is to describe them all. Although the existence of such functions was known long ago, the classification of them is new, and our proof of this classification is rather indirect and unexpected.

Proof of Proposition 5.2. Writing $k := \deg(g)$, we have $|g^{-1}(\alpha_i)_{\text{set}}| \leq k/e_i$, so Lemma 5.4 implies that

$$2k - 2 \geq \sum_{i=1}^{3} \bigl(k - |g^{-1}(\alpha)_{\text{set}}|\bigr) \geq \sum_{i=1}^{3} \Bigl(k - \frac{k}{e_i}\Bigr),$$

whence $\sum_{i=1}^{3} 1/e_i > 1$. Since the e_i's are integers greater than 1, and since $1 = 1/3 + 1/3 + 1/3 = 1/2 + 1/4 + 1/4 = 1/2 + 1/3 + 1/6$, it follows that (e_1, e_2, e_3) is a permutation of an element of $\mathcal{N}$. Now let π, N, μ be as in the statement of the result. It is easy to check directly that for $\gamma \in \mathbb{C}_\infty$ the multiplicity of f_N at γ is $N(1)$, $N(2)$, $N(3)$, or 1, according as $f_N(\gamma)$ is 1, 0, ∞, or another value. Now view f_N and $\widehat{g} := \mu \circ g$ as branched coverings $S^2 \to S^2$, and let B be the set of branch points of $\widehat{g}$, which includes the branch points of f_N. Then the branched coverings f_N and $\widehat{g}$ become topological covering maps when we restrict the domain to avoid preimages of B, yielding finite topological covering maps $\psi : S^2 \setminus f_N^{-1}(B) \to S^2 \setminus B$ and $\phi : S^2 \setminus \widehat{g}^{-1}(B) \to S^2 \setminus B$. Form the pullback of ϕ along ψ as usual, yielding the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \pi_2\ } & S^2 \setminus f_N^{-1}(B) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \psi} \\ S^2 \setminus \widehat{g}^{-1}(B) & \xrightarrow{\ \phi\ } & S^2 \setminus B \end{array}$$

where $X := \{(a, b) \in (S^2 \setminus \widehat{g}^{-1}(B)) \times (S^2 \setminus f_N^{-1}(B)) \colon \phi(a) = \psi(b)\}$ and π_1 and π_2 are projections on the first and second coordinates, respectively. We may compactify the topological covering map $\phi \circ \pi_1 : X \to S^2 \setminus B$ (see e.g. [10, §2]) in order to obtain a branched covering $\eta : \widehat{X} \to S^2$ which factors as $\eta = g \circ \widehat{\pi_1} = f_N \circ \widehat{\pi_2}$ where $\widehat{\pi_i}$ is the induced extension of π_i. For each $\beta \in S^2$, the multiplicity under $\widehat{g}$ of every point in $\widehat{g}^{-1}(\beta)$ is divisible by the multiplicity under f_N of every point in $f_N^{-1}(\beta)$, so by elementary covering space theory it follows that $\widehat{\pi_1}$ is an unbranched covering. Since S^2 is simply connected, this implies that the restriction of $\widehat{\pi_1}$ to any connected component Y of $\widehat{X}$ will be a homeomorphism $\theta_1 \colon Y \to S^2$, so if θ_2 is the restriction of $\widehat{\pi_2}$ to Y then $\widehat{g} = f_N \circ \theta_2 \circ \theta_1^{-1}$. Here $\theta_2 \circ \theta_1^{-1}$ is a finite-degree branched covering $S^2 \to S^2$. Of course, any such branched covering induces a holomorphic function $\mathbb{C}_\infty \to \mathbb{C}_\infty$, which in turn is a rational function $h(x)$ such that $\mu \circ g = f_N \circ h$. Finally, the remaining assertions about the factorization of $f_N(x) - f_N(y)$ and the orders of elements of T_N are easy to verify directly, given that T_N is the group (under the operation of functional composition) generated by the set U_N defined as follows:

$$U_{(2,2,r)} := \{\zeta x^e \colon \zeta^r = 1,\, e \in \{1, -1\}\}$$

$$U_{(2,3,3)} := \left\{e^{2\pi i/3}x,\, \frac{x+2}{x-1}\right\}$$

$$U_{(2,3,4)} := \left\{ix,\, \frac{x+1}{x-1}\right\}$$

$$U_{(2,3,5)} := \left\{\zeta x,\, \frac{(\zeta^3+1)x+1}{x-\zeta^2-1}\right\} \quad \text{where } \zeta := e^{2\pi i/5}. \qquad \square$$

Remark 5.5. The topological argument in the above proof can be written in the language of algebraic geometry, by considering the normalizations of components of the fibered product of the morphisms $\mathbb{P}^1 \to \mathbb{P}^1$ induced by f_N and $\mu \circ g$. We chose topological language since we thought this would be more familiar to some complex analysts in our audience.

6. Minimal shared multisets

In this section we prove a generalization of Theorem 1.6, by describing the collection of all shared multisets for quasi-equivalent nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$. We begin by addressing the analogous question for meromorphic functions on an arbitrary complex manifold.

6.1. **Arbitrary complex manifolds.**

Definition 6.1. For any complex manifold M and any nonconstant $p, q \in \mathcal{M}(M)$, a *minimal shared multiset* for p and q is a nonempty finite multiset S of elements of $\mathbb{C}_\infty$ such that S is shared CM by p and q, but no nonempty proper sub-multiset of S is shared CM by p and q.

Lemma 6.2. *If S is a finite multiset of elements of $\mathbb{C}_\infty$, then S is shared CM by p and q if and only if S is the union of finitely many minimal shared multisets for p and q.*

Proof. If S is shared, we may assume S is not. and T is a minimal shared multiset contained in S, then $S \setminus T$ is a shared multiset which is smaller than S, so by induction on $|S|$ we see that S is a union of minimal shared multisets. Conversely, any union of shared multisets is itself shared. $\square$

In light of the above result, in order to describe all shared multisets for p and q, it suffices to describe the minimal shared multisets. We now introduce a large collection of shared multisets T_α in case $g(p) = g(q)$ for some nonconstant $g \in \mathbb{C}(x)$. It will turn out that, in many situations, these T_α comprise the collection of all minimal shared multisets.

Definition 6.3. Let p and q be quasi-equivalent nonconstant meromorphic functions on a complex manifold M and let $g \in \mathbb{C}(x)$ be their coequalizer. For any $\alpha \in \mathbb{C}_\infty$, let R_α be the multiset $g^{-1}(\alpha)$, let $\operatorname{gcdmult}(R_\alpha)$ denote the greatest common divisor of the multiplicities of the elements of R_α, and let

T_α be the multiset having the same underlying set as R_α, but in which the multiplicity of each element is $1/\operatorname{gcdmult}(R_\alpha)$ times the multiplicity of the element in R_α.

Example 6.4. If $p = -q \in \mathcal{M}(M) \setminus \mathbb{C}$ then x^2 is a coequalizer, so that $R_0 = \{0, 0\}$ has $\operatorname{gcdmult}(R_0) = 2$ and thus $T_0 = \{0\}$; likewise $T_\infty = \{\infty\}$, but for any $\alpha \notin \{0, \infty\}$ we have $R_\alpha = \{\beta, -\beta\}$ with $\beta^2 = \alpha$, so that $\operatorname{gcdmult}(R_\alpha) = 1$ and $T_\alpha = R_\alpha$.

Lemma 6.5. *Let p, q be nonconstant meromorphic functions on a complex manifold M such that $\mathcal{G}_1$ is nonempty. Then each T_α with $\alpha \in \mathbb{C}_\infty$ is a nonempty finite multiset which is shared CM by p and q, and every minimal shared multiset is contained in one of the multisets T_α. The collection of all T_α's depends only on p and q, and not on the choice of a minimal-degree function in $\mathcal{G}_1$.*

Proof. By taking preimages of α on both sides of the equation $g \circ p = g \circ q$, we see that p, q share CM R_α, and hence also T_α. Plainly T_α is nonempty and finite. By Proposition 4.2, any other choice of g has the form $\widehat{g} := \mu \circ g$ for some Möbius transformation μ; denoting the corresponding multisets by $\widehat{T}_\alpha$, it follows that $T_\alpha = \widehat{T}_{\mu(\alpha)}$, so that the collection of all T_α's equals the collection of all $\widehat{T}_\alpha$'s. Finally, the union of the T_α's is $\mathbb{C}_\infty$, so for any minimal shared multiset S there is some α for which $S \cap T_\alpha$ is nonempty; but then $S \cap T_\alpha$ is a shared multiset, so minimality of S implies $S \cap T_\alpha = S$, whence $S \subseteq T_\alpha$. □

We now show that if M is a compact complex manifold and $\mathcal{G}_1$ is nonempty then the T_α comprise all minimal shared multisets for p and q.

Proposition 6.6. *If p and q are nonconstant meromorphic functions on a compact Riemann surface $\mathcal{R}$, and $\mathcal{G}_1$ is nonempty, then the minimal shared multisets for p and q are precisely the multisets T_α with $\alpha \in \mathbb{C}_\infty$.*

Proof. Pick a minimal-degree $g \in \mathcal{G}_1$, and suppose that some T_α is not a minimal shared multiset. Since T_α is shared CM by p and q, it is the union of two or more (not necessarily distinct) minimal shared multisets. Since $\operatorname{gcdmult}(T_\alpha) = 1$, these minimal shared multisets in T_α cannot all be equal, so T_α contains two disjoint minimal shared multisets S_1 and S_2. By Lemma 3.5, there are integers $n_1, n_2 > 0$, a nonconstant $h \in \mathbb{C}(x)$, and a $\gamma \in \mathbb{C}^*$ such that $h \circ p = \gamma h \circ q$. For any $\beta \in \mathbb{C}_\infty$ with $\beta \neq \alpha$, the set $h(T_\beta)_{\text{set}}$ is a nonempty finite subset of $\mathbb{C}^*$, and for any $\delta \in T_\beta$ there is some $\epsilon \in M$ such that $\delta = q(\epsilon)$, whence $\delta' := p(\epsilon)$ is an element of T_β satisfying

$$h(\delta') = h(p(\epsilon)) = \gamma \cdot h(q(\epsilon)) = \gamma \cdot h(\delta).$$

Thus $h(T_\beta)_{\text{set}}$ is preserved by multiplication by γ, so γ is a root of unity and hence $h^n(p) = h^n(q)$ for some positive integer n. By Proposition 4.2 we have $h^n = d \circ g$ for some $d \in \mathbb{C}(x)$, so the divisor of h^n is a $\mathbb{Z}$-linear combination of $g^{-1}(\alpha)$'s. But this is impossible because the divisor of h^n

has positive coefficients at the elements of S_1 and negative coefficients at the elements of S_2. This contradiction shows that in fact every T_α must be a minimal shared multiset. $\square$

Corollary 6.7. *For any multiset S shared by nonconstant quasi-equivalent meromorphic functions p and q on a compact Riemann surface $\mathcal{R}$ with coequalizer $g \in \mathbb{C}(x) \setminus \mathbb{C}$, there exists an integer m and a multiset A for which $S^m = g^{-1}(A)$.*

Proof. By Lemma 6.2 and Proposition 6.6, the multiset S is of the form $\bigcup_{\alpha \in I} T_\alpha$ for a multiset I consisting of elements of $\mathbb{C}_\infty$. Letting $n_\alpha :=$ gcdmult$(g^{-1}(\alpha))$ and $m = \prod_{\alpha \in I} n_\alpha$ we see that $S^m = g^{-1}\left(\bigcup_{\alpha \in I} \{\alpha\}^{m/n_\alpha}\right)$. $\square$

6.2. **Complex m-space.** We now prove the following generalization of Theorem 1.6, which involves both the shared multisets T_α from Definition 6.3 and the set $\mathcal{G}_1$ from Definition 4.1.

Theorem 6.8. *Pick nonconstant quasi-equivalent $p, q \in \mathcal{M}(\mathbb{C}^m)$ and let $g(x)$ be their coequalizer. Then one of the following occurs:*

(6.8.1) *The collection of all multisets T_α with $\alpha \in \mathbb{C}_\infty$ equals the collection of all minimal shared multisets for p and q.*

(6.8.2) *For some $\beta \in \mathbb{C}_\infty$, the multiset T_β is the union of positive numbers of copies of each of two distinct minimal shared multisets S_1, S_2, and the collection of all minimal shared multisets consists of S_1, S_2, and all T_α with $\alpha \neq \beta$. In this case we can write $p = p_0 \circ r$ and $q = q_0 \circ r$ for some $r \in \mathcal{M}(\mathbb{C}^m)$ and some $p_0, q_0 \in \mathbb{C}(x)$ such that $g(p_0) = g(q_0)$, and for any such p_0, q_0, r there will be two Picard exceptional values γ, δ of r, with $\gamma \in p_0^{-1}(S_1) \cap q_0^{-1}(S_2)$ and $\delta \in p_0^{-1}(S_2) \cap q_0^{-1}(S_1)$, where in addition for each $i = 1, 2$ the multisets $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ coincide except for copies of γ and δ.*

Proof. Since $g(p) = g(q)$, the functions p and q are algebraically independent. By Lemma 3.7, there is a compact Riemann surface $\mathcal{R}$ for which $p = p_0(r)$ and $q = q_0(r)$ for some $p_0, q_0 \in \mathcal{M}(\mathcal{R})$ and some holomorphic map $r\colon \mathbb{C}^m \to \mathcal{R}$. Thus for any multiset S of elements of $\mathbb{C}_\infty$, the multiset $p^{-1}(S)$ is the union of $r^{-1}(\alpha)$ for $\alpha \in p_0^{-1}(S)$. It follows that p and q share S CM if and only if the multiset differences $A(S) := p_0^{-1}(S) \setminus q_0^{-1}(S)$ and $B(S) := q_0^{-1}(S) \setminus p_0^{-1}(S)$ both consist of elements of the set $\mathcal{E} := \mathcal{R} \setminus r(\mathbb{C}^m)$, which has size at most 2 by Lemma 3.10.

Suppose (6.8.1) does not hold, so, by Lemma 6.5, some T_α is not a minimal shared multiset. Since gcdmult$(T_\alpha) = 1$, it follows that T_α contains two disjoint minimal shared multisets S_1 and S_2. The identity $g(p_0(r)) = g(p) = g(q) = g(q_0(r))$ implies that $g(p_0) = g(q_0)$, so in particular $\deg(p_0) = \deg(q_0)$. Thus $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ have the same size, so also $A_i := A(S_i)$ and $B_i := B(S_i)$ have the same size n_i. Proposition 6.6 implies that S_i is not shared by p_0 and q_0, so $n_i > 0$. Thus A_i contains an element γ_i. Since

A_i consists of elements of $\mathcal{E}$, and $|\mathcal{E}| \le 2$, disjointness of the S_i's implies that A_i consists of n_i copies of γ_i. Likewise, since A_i and B_i are disjoint, B_i must consist of n_i copies of γ_{3-i}, so (6.8.2) holds. □

Example 6.9. The second possibility in Theorem 6.8 can actually occur. For instance, let k, n be integers with $0 < k < n$, put $\zeta := e^{2\pi i/n}$, and let $p := (e^x + \zeta^k)/(e^x + 1)$ and $q := \zeta p$. Then we may choose $g := x^n$, so that $g^{-1}(1) = \{1, \zeta, \zeta^2, \dots, \zeta^{n-1}\}$ is the union of $S_1 := \{\zeta, \zeta^2, \dots, \zeta^k\}$ and $S_2 := \{\zeta^{k+1}, \zeta^{k+2}, \dots, \zeta^n\}$. Here p has no preimages of 1 or ζ^k, so q has no preimages of ζ or ζ^{k+1}, whence

$$p^{-1}(S_1) = p^{-1}(\{\zeta, \zeta^2, \dots, \zeta^{k-1}\}) = q^{-1}(\{\zeta^2, \zeta^3, \dots, \zeta^k\}) = q^{-1}(S_1),$$

and likewise $p^{-1}(S_2) = q^{-1}(S_2)$.

7. Bounding the degree of a rational function relating p and q

In this section we use the results of the previous two sections in order to bound the degree of a minimal-degree element of $\mathcal{G}_1$ in terms of the sizes of shared multisets. The combination of these bounds with Theorem 3.1 yields Theorem 1.2.

Theorem 7.1. *Pick nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$, and let $S_1, \dots, S_n$ be finite multisets of elements of $\mathbb{C}_\infty$ such that p, q share CM each S_i, where $n \ge 4$ and no S_i is contained in the union of the other S_j's. Then $g(p) = g(q)$ for some nonconstant $g \in \mathbb{C}(x)$ such that $\deg(g) \le \frac{1}{n-3}(-2 + \sum_{i=1}^n |(S_i)_{set}|)$. If $n \ge 5$ then we can choose $g(x)$ to have degree at most $\max_i |S_i|$. Moreover, if $n \ge 5$ and the S_i's are minimal shared multisets then $\max_i |S_i|$ is the smallest degree of any nonconstant $g \in \mathbb{C}(x)$ for which $g(p) = g(q)$.*

Remark 7.2. In order to obtain the best bound on $\deg(g)$ from Theorem 7.1, it is sometimes advantageous to ignore some of the S_i's when applying the bounds in this result. For instance, if $n > 5$ then we can choose $g(x)$ to have degree at most the size of the fifth-smallest S_i. Likewise, if $n = 5$ and one S_i is much larger than the others then the best bound will come from applying the first bound in Theorem 7.1 to the other four S_j's. This shows that the first bound is sometimes better than the second bound when they both apply; conversely, if $n \ge 5$ and the S_i's are sets of the same size then the second bound is better than the first.

Proof. By Theorem 3.1 there is a nonconstant $g \in \mathbb{C}(x)$ such that $g(p) = g(q)$. Choose one such $g(x)$ for which $k := \deg(g)$ is as small as possible. For each i, let R_i be a minimal shared multiset contained in $S_i \setminus \cup_{j \ne i} S_j$, so that the R_i's are pairwise disjoint. Let I be the set of values i for which R_i has the form T_{α_i} with $\alpha_i \in \mathbb{C}_\infty$. Theorem 6.8 implies that $|I| \ge n - 2$, and in addition if $|I| = n - 2$ then there is some $\alpha \in \mathbb{C}_\infty$ for which T_α is the union of copies of the two multisets R_i with $i \notin I$. Thus if $|I| = n - 2$ then

$V := g(\cup_{i=1}^n R_i)$ has size $n-1$, and $g^{-1}(V)$ is the union of copies of the R_i's, so Lemma 5.4 yields

$$\begin{aligned} 2k-2 &\geq \sum_{\alpha\in V}\bigl(k - |g^{-1}(\alpha)_{\text{set}}|\bigr) \\ &= (n-1)k - \sum_{i=1}^n |(R_i)_{\text{set}}| \\ &\geq (n-1)k - \sum_{i=1}^n |(S_i)_{\text{set}}|, \end{aligned}$$

whence $k \leq \frac{1}{n-3}\bigl(-2+\sum_{i=1}^n |(S_i)_{\text{set}}|\bigr)$. If $|I|\geq n-1$ then $V := g(\cup_{i\in I} R_i)$ has the same size as I, so Lemma 5.4 yields

$$\begin{aligned} 2k-2 &\geq \sum_{\alpha\in V}\bigl(k - |g^{-1}(\alpha)_{\text{set}}|\bigr) \\ &= k|I| - \sum_{i\in I} |(R_i)_{\text{set}}| \\ &\geq (n-1)k - \sum_{i=1}^n |(S_i)_{\text{set}}|, \end{aligned}$$

so that again $k \leq \frac{1}{n-3}\bigl(-2+\sum_{i=1}^n |(S_i)_{\text{set}}|\bigr)$.

By Proposition 5.1 there are at most two elements $i\in I$ for which $g^{-1}(\alpha_i)$ consists of more than one copy of R_i. Thus if $n\geq 5$ then, since $|I|\geq n-2\geq 3$, there is some $i\in I$ for which $g^{-1}(\alpha_i)=R_i$, so that

$$k = \deg(g) = |g^{-1}(\alpha_i)| = |R_i| \leq \max(\{|S_j| \colon 1\leq j\leq n\}).$$

Finally, if $n\geq 5$ then $k=|R_i|$ for some i, and every R_j is contained in some $g^{-1}(\alpha_j)$ and hence has size at most k, so $k=\max_{j=1}^n |R_j|$. □

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Cranbrook Kingswood Upper School, 550 Lone Pine Road, Bloomfield Hills, MI 48304 USA

Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109-1043 USA

Email address: zieve@umich.edu

URL: http://www.math.lsa.umich.edu/∼zieve/