

Lebesgue Measure Preserving Thompson's Monoid

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Abstract

This paper defines Lebesgue measure preserving Thompson's monoid, denoted by \mathbb{G} , which is modeled on Thompson's group \mathbb{F} except that the elements of \mathbb{G} are non-invertible. Moreover, it is required that the elements of \mathbb{G} preserve Lebesgue measure. Monoid \mathbb{G} exhibits very different properties from Thompson's group \mathbb{F} . The paper studies a number of algebraic (group-theoretic) and dynamical properties of \mathbb{G} including approximation, mixing, periodicity, entropy, decomposition, generators, and topological conjugacy.

1 Introduction

In this paper we define Lebesgue measure preserving Thompson's monoid and study its algebraic and dynamical properties. This study is at an intersection of two subjects of research.

The first subject is concerned with Lebesgue measure preserving interval maps of $[0, 1]$ onto itself, which studies dynamical properties such as transitivity, mixing, periodic points and metric entropy and finds important applications in the abstract formulation of dynamical systems, chaos theory and ergodic theory [1, 2]. The author in [1] motivates the study of interval maps by stating that the "most interesting" part of some higher-dimensional systems can be of lower dimensions, which allows, in some cases, to boil down to systems in dimension one. In particular, a recent paper [3] studies a special form of interval maps, namely, piecewise affine maps.

The second subject is concerned with Thompson's group \mathbb{F} [4, 5], which is the group of piecewise affine maps from $[0, 1]$ onto itself whose derivatives are integer powers of 2 and points at which the derivatives are discontinuous are dyadic numbers. As the derivatives are always positive, the orientation of maps is preserved. Thompson group \mathbb{F} has a collection of unusual algebraic properties that make it appealing in many different and diverse areas of mathematics such as group theory, combinatorics [6] and cryptography [7].

Except for the identity map, any Thompson's group \mathbb{F} map does not preserve Lebesgue measure and any Lebesgue measure preserving interval map does not preserve orientation and thus not belong to Thompson's group \mathbb{F} . Thus these two subjects do not naturally intersect. We intend to build on Thompson's group \mathbb{F} by making important changes to preserve Lebesgue measure. More precisely, we define Lebesgue measure preserving Thompson's monoid, denoted by \mathbb{G} . Monoid \mathbb{G} is similar to \mathbb{F} except that the derivatives of piecewise affine maps can be negative. As a result, the maps in \mathbb{G} are non-invertible except for some trivial maps and exhibit very different properties from those in \mathbb{F} .

To the best of our knowledge, Lebesgue measure preserving Thompson's monoid has not been proposed or studied in the literature. Unless explicitly mentioned, all the results presented and proved in this paper are original.

The goal of this paper is to study unique properties of \mathbb{G} . The main results of this paper are summarized as follows.

- We show that any continuous measure preserving map can be approximated by a map in \mathbb{G} with any required precision. Moreover, we show that the approximating map in \mathbb{G} can be locally eventually onto (LEO) and achieve any target value of entropy that is at least 2.
- We show that for any element of \mathbb{G} , topological mixing (TM) is equivalent to LEO and every dyadic point is preperiodic. Thus, any map in \mathbb{G} is Markov. We show that for maps in a subset of \mathbb{G} there exist periodic points with period of 3, an essential feature of chaotic maps. We characterize periods of periodic points of other maps in \mathbb{G} .
- We show that unlike \mathbb{F} , \mathbb{G} is not finitely generated. We define equivalence classes for maps in \mathbb{G} and construct a monoid by sets of equivalence classes such that the monoid is finitely generated and any map in \mathbb{G} is an element of an equivalence class in the monoid.
- We derive sufficient conditions for a continuous map to be topologically conjugate to a measure preserving piecewise affine continuous map and in particular a map in \mathbb{G} .

The main results of this paper improve several results of [3]. For example, we show that \mathbb{G} that is both LEO and Markov is dense in the set of continuous measure preserving maps. Because \mathbb{G} is a subset of piecewise affine continuous measure preserving maps, this result is stronger than [3, Proposition. 7], which shows that piecewise affine continuous measure preserving maps that are both LEO and Markov is dense in the set of continuous measure preserving maps.

At an intersection of these two subjects of research, the paper demonstrates an interesting interplay between algebraic (group-theoretic) and dynamical settings. For example, in general, LEO implies TM and the converse does not hold; however, we show that for any element of \mathbb{G} , TM is equivalent to LEO and any map in \mathbb{G} is Markov. As another example, we show that the algebraic structure of \mathbb{G} leads to a simple characterization of periods of periodic points of maps in \mathbb{G} and allows the use of Markov partition to study measure preserving topological conjugate maps.

The remainder of the paper is organized as follows. Section 2 reviews the basic properties of measure preserving interval maps and Thompson's group \mathbb{F} and defines measure preserving Thompson's monoid \mathbb{G} . Section 3 shows that a map in \mathbb{G} can approximate any continuous measure preserving map with any required precision. While locally eventually onto (LEO) implies topological mixing (TM) for any maps, Section 4 shows that TM is equivalent to LEO for any element of \mathbb{G} and that a map in \mathbb{G} that is LEO can approximate any continuous measure preserving map with any required precision. Section 5 shows a salient feature of \mathbb{G} that every dyadic point is preperiodic. As a result, any map in \mathbb{G} is Markov. Section 5 furthermore characterizes the periods of periodic points of maps in \mathbb{G} . Section 6 investigates the entropy properties of \mathbb{G} and shows that any entropy greater than or equal to 2 can be achieved by \mathbb{G} . Section 7 shows that any map in \mathbb{G} can be expressed as a composition of a finite number of basic maps in \mathbb{G} and the generators in \mathbb{F} . Section 8 shows that unlike \mathbb{F} , \mathbb{G} is not finitely generated. Section 8 defines the notions of equivalence classes and sets of equivalence classes, constructs a monoid of sets of equivalence classes and shows that the monoid has a finite number of generators and that any map in \mathbb{G} is an element of an equivalence class in the monoid. Section 8 furthermore introduces a metric to characterize equivalence classes. Section 9 studies topological conjugacy under the measure preservation constraint and uses Markov partition to characterize continuous maps that are conjugate to measure preserving maps. Finally, Section 10 proposes a few areas for future study.

2 Basic Definitions and Properties

2.1 Notations

Consider continuous interval maps from $[0, 1]$ to $[0, 1]$. Let h_1 and h_2 be two maps. Denote by $h_1 \circ h_2$ the composition of h_1 and h_2 where $h_1 \circ h_2(x) = h_1(h_2(x))$. The composition of more than two maps can be recursively defined with this definition. For any $y \in [0, 1]$, define $h^{-1}(y) = \{x \in [0, 1] : h(x) = y\}$.

Interval map h defines a topological dynamical system whose evolution is given by successive iterations of the map. For any positive integer n , $h^n = \underbrace{h \circ h \circ \dots \circ h}_{n \text{ times}}$. By convention, h^0 is the identity map. Point x is *preperiodic* if positive integers $n > m$ exist such that $h^n(x) = h^m(x)$. If $m = 0$, then x is *periodic*.

Define trivial maps $g_{0,+}(x) = x$ and $g_{0,-}(x) = 1 - x$ for $x \in [0, 1]$.

Let A be a point in the plane of $[0, 1] \times [0, 1]$. Denote by A_x and A_y the x - and y -coordinates of point A , respectively. If A is on the graph of map h , $A_y = h(A_x)$.

Let \mathcal{I} be a interval in $[0, 1]$. Let \mathcal{I}° represent the interior of \mathcal{I} . The left and right endpoints of \mathcal{I} are denoted by $\mathcal{I}^0, \mathcal{I}^1$, respectively. If \mathcal{I} is closed, then $\mathcal{I} = [\mathcal{I}^0, \mathcal{I}^1]$. Let $|\mathcal{I}|$ represent the measure of the interval: $|\mathcal{I}| = \mathcal{I}^1 - \mathcal{I}^0$. For two distinct intervals \mathcal{I}_1 and \mathcal{I}_2 , $\mathcal{I}_1 < \mathcal{I}_2$ if $x_1 \leq x_2, \forall x_1 \in \mathcal{I}_1, x_2 \in \mathcal{I}_2$.

Let \mathcal{I}, \mathcal{J} be two closed intervals of $[0, 1]$ and f_1, f_2 be two maps. Let $f_1(\mathcal{I}) \simeq f_2(\mathcal{J})$ if f_2 can be linearly transformed from f_1 . That is, if $x_1 = \mathcal{I}^0 + \alpha(\mathcal{I}^1 - \mathcal{I}^0)$, and $x_2 = \mathcal{J}^0 + \alpha(\mathcal{J}^1 - \mathcal{J}^0)$ for some $\alpha \in [0, 1]$, then $f_1(x_1) = f_2(x_2)$. When f_2 is a trivial map, $f_1(\mathcal{I}) \simeq \mathcal{J}$ if f_1 is an affine map.

A set of distinct closed intervals $\{\mathcal{I}_1, \dots, \mathcal{I}_n\}$ is a partition of $[0, 1]$ if $\mathcal{I}_i^\circ \cap \mathcal{I}_j^\circ = \emptyset$ for any $i \neq j$ and $\bigcup_{i=1}^n \mathcal{I}_i = [0, 1]$. If $\mathcal{I}_1 < \dots < \mathcal{I}_n$, then $\{|\mathcal{I}_i|\}$ completely determines $\{\mathcal{I}_i\}$. A subset of $\{\mathcal{I}_i\}$ may be a single point, i.e., $\mathcal{I}_j^0 = \mathcal{I}_j^1$ where some j .

Denote by $\langle a, b \rangle$ interval $[a, b]$ if $a \leq b$ and interval $[b, a]$ if $b < a$.

2.2 λ -Preserving Interval Maps

Denote by λ the Lebesgue measure on $[0, 1]$ and \mathcal{B} all Borel sets on $[0, 1]$.

Definition 1 (*λ -Preserving Interval Maps*). Continuous interval map h is λ -preserving if $\forall A \in \mathcal{B}, \lambda(A) = \lambda(h^{-1}(A))$.

Remark. Definition 1 does not imply $\lambda(A) = \lambda(h(A))$ for λ -preserving h . In fact, one can easily show that if h is λ -preserving, $\lambda(A) \leq \lambda(h(A))$ for any $A \in \mathcal{B}$. Except for the trivial maps of $g_{0,+}$ and $g_{0,-}$, h is not invertible and $\exists A \in \mathcal{B}$ such that $\lambda(A) < \lambda(h(A))$.

For simplicity, $\lambda(A)$ is also written as $|A|$.

Let $C(\lambda)$ be the set of all continuous λ -preserving interval maps. For each map $h \in C(\lambda)$, the set of periodic points is dense on $[0, 1]$ because of the Poincaré Recurrence Theorem and the fact that the closures of recurrent points and periodic points coincide [3].

Let $PA(\lambda)$ be the subset of $C(\lambda)$ consisting of all piecewise affine maps.

2.3 Thompson's Group \mathbb{F}

Thompson's group \mathbb{F} has a few different representations such as group presentations, rectangle diagrams and piecewise linear homeomorphisms. The following focuses on the representation of piecewise

linear homeomorphisms because it is closely related to λ -preserving Thompson's monoid to be introduced in the next section.

Definition 2 (*Thompson's Group \mathbb{F}*). A homeomorphism f from $[0, 1]$ onto $[0, 1]$ is an element of Thompson's group \mathbb{F} if

- f is piecewise affine;
- f is differentiable except at finitely many points;
- The x -coordinate of each of these points of non-differentiability is a dyadic number, i.e., a rational number whose denominator is an integer power of 2;
- On the intervals where f is differentiable, the derivatives are integer powers of 2.

In the remainder of this paper, f is referred to an element in Thompson's group \mathbb{F} .

Remark. It is easily to see that $f(0) = 0, f(1) = 1$ and f is strictly increasing on $[0, 1]$ and is thus invertible. Except for the trivial map of $f = g_{0,+}$, f is not λ -preserving.

Example 1. Define the following two maps in \mathbb{F} .

$$f_A(x) = \begin{cases} \frac{x}{2}, & 0 \leq x \leq \frac{1}{2}, \\ x - \frac{1}{4}, & \frac{1}{2} \leq x \leq \frac{3}{4}, \\ 2x - 1, & \frac{3}{4} \leq x \leq 1, \end{cases} \quad f_B(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2}, \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \leq x \leq \frac{3}{4}, \\ x - \frac{1}{8}, & \frac{3}{4} \leq x \leq \frac{7}{8}, \\ 2x - 1, & \frac{7}{8} \leq x \leq 1. \end{cases} \quad (1)$$

The significance of f_A and f_B is that Thompson's group \mathbb{F} is generated by the two maps. That is, any $f \in \mathbb{F}$ can be represented by a composition of possibly multiple f_A and f_B in certain order [4].

2.4 λ -Preserving Thompson's Monoid \mathbb{G}

Definition 3 (*λ -Preserving Thompson's Monoid \mathbb{G}*). Continuous interval map g from $[0, 1]$ onto $[0, 1]$ is an element of λ -preserving Thompson's monoid \mathbb{G} if

- g is λ -preserving;
- g is piecewise affine;
- g is differentiable except at finitely many points;
- The x -coordinate of each of these points of non-differentiability is a dyadic number;
- On an interval where g is differentiable, the derivative is positive or negative and the absolute value of the derivative is an integer power of 2.

Remark. The difference between \mathbb{G} and \mathbb{F} is that the derivatives can be negative in the maps of \mathbb{G} , which makes it possible for them to be λ -preserving.

Remark. It is easy to see that if $g_1, g_2, g_3 \in \mathbb{G}$, then $g_1 \circ g_2 \in \mathbb{G}$ and $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$. Trivial map $g_{0,+}$ is the identity element of \mathbb{G} . However, an inverse may not always exist for any given $g \in \mathbb{G}$. This is the reason that the set of maps satisfying these conditions is a monoid.

Remark. $\mathbb{G} \subset PA(\lambda)$. From Section 2.2, the set of periodic points of $g \in \mathbb{G}$ is dense on $[0, 1]$.

In the remainder of this paper, g is referred to an element in λ -preserving Thompson's monoid \mathbb{G} . When g is an affine segment on an interval, for simplicity, refer the derivative of g on the interval to as the slope of the affine segment.

Definition 4 (*Breakpoints*). Let $g \in \mathbb{G}$. A breakpoint of g is either an endpoint at $x = 0$ or $x = 1$ or a point at which the derivative of g is discontinuous. A breakpoint that is not an endpoint is referred to

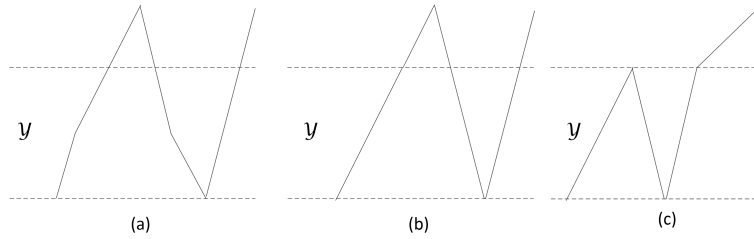


Figure 1: Illustration of the definitions of legs (a), affine legs (b) and window perturbation (c). $m = 3$ in the figure.

as interior breakpoint. An interior breakpoint is further categorized into type I and type II. At a type I breakpoint, the left and right derivatives are of the same sign. At a type II breakpoint, the left and right derivatives are of the opposite signs.

Point (x, y) is said to be dyadic if both x and y are dyadic.

Lemma 1. For any point (x, y) of $g \in \mathbb{G}$, y is dyadic if and only if x is dyadic.

Proof. Let (x_0, y_0) be a breakpoint of g . By definition, x_0 is dyadic. If $y_0 = 1$, then (x_0, y_0) is already dyadic. Otherwise, let $c \in g^{-1}(1)$. Suppose $c > x_0$. (The case of $c < x_0$ can be proven analogously.) Let $x_0 < x_1 < \dots < x_n = c$ be the set of breakpoints between x_0 and c and the slope of the affine segment on $[x_{i-1}, x_i]$ be $(-1)^{p_i} 2^{k_i}$, with p_i equal to 0 or 1 and k_i an integer, for $i = 1, \dots, n$. Thus, $g(x_n) - g(x_0) = 1 - y_0 = \sum_{i=1}^n (-1)^{p_i} 2^{k_i} (x_i - x_{i-1})$, which is a dyadic number. Hence, (x_0, y_0) is dyadic. Therefore, any breakpoint of g is dyadic. The lemma follows immediately because both endpoints of an affine segment are dyadic and the derivative is in the form of $\pm 2^k$ with integer k . \square

Lemma 2. For $y \in [0, 1]$, suppose that $g^{-1}(y) = \{x_1, \dots, x_n\}$ and none of x_1, \dots, x_n are breakpoints. Map g is λ -preserving if and only if

$$\sum_{i=1}^n \frac{1}{|g'(x_i)|} = \sum_{i=1}^n 2^{-k_i} = 1, \tag{2}$$

where k_i is integer and $|g'(x_i)| = 2^{k_i}$ is the absolute value of the slope of the affine segment on which x_i resides.

Proof. Let $\mathcal{Y} = [y - \delta, y + \delta]$ for $\delta > 0$. For a sufficiently small δ , $g^{-1}(\mathcal{Y}) = \bigcup_{i=1}^n \mathcal{I}_i$, where intervals \mathcal{I}_i are disjoint, $x_i \in \mathcal{I}_i$, and $g(\mathcal{I}_i) = \mathcal{Y}$ for $i = 1, \dots, n$. $\lambda(\mathcal{Y}) = \lambda(g(\mathcal{I}_i)) = |g'(x_i)| \lambda(\mathcal{I}_i)$ as $\delta \rightarrow 0$. By λ -preservation and because \mathcal{I}_i are disjoint, $\lambda(\mathcal{Y}) = \lambda(g^{-1}(\mathcal{Y})) = \sum_{i=1}^n \lambda(\mathcal{I}_i)$. (2) follows immediately. \square

To satisfy (2), k_i must be non-negative for any i . In contrast, for f , a derivative can be a negative integer power of 2. Moreover, if $n > 1$, $g'(x_i)$ has alternating signs: $g'(x_i)g'(x_{i+1}) < 0$ for $i = 1, \dots, n - 1$. Unlike f , g is not orientation-preserving except for the trivial maps.

Definition 5 (Legs and Affine Legs). Let interval $\mathcal{Y} \subset [0, 1]$. If except for a finite number, $\forall y \in \mathcal{Y}$, set $g^{-1}(y)$ has m elements, then $g^{-1}(\mathcal{Y})$ is said to have m legs. When $g^{-1}(\mathcal{Y})$ has m legs, m intervals $\mathcal{I}_1, \dots, \mathcal{I}_m$ with mutually disjoint interiors exist such that $g^{-1}(\mathcal{Y}) = \bigcup_{i=1}^m \mathcal{I}_i$, and g is monotone on every \mathcal{I}_i and $\mathcal{Y} = g(\mathcal{I}_i)$ for any i . The graph of g on \mathcal{I}_i is referred to as the i -th leg. If g is affine on every \mathcal{I}_i , then $g^{-1}(\mathcal{Y})$ is said to have m affine legs.

Definition 6 (Window Perturbation). When $g^{-1}(\mathcal{Y})$ has m affine legs for interval \mathcal{Y} , if $\bigcup_{i=1}^m \mathcal{I}_i$ is an interval \mathcal{I} , g is said to be an m -fold window perturbation on \mathcal{I} .

Figure 1 illustrates the definitions of legs, affine legs and window perturbation.

3 Approximation

Define metric ρ by $\rho(h_1, h_2) = \sup_{x \in [0,1]} |h_1(x) - h_2(x)|$ for any two continuous interval maps h_1 and h_2 . This section will show that \mathbb{G} has rich approximation capability in the sense that any λ -preserving continuous map h can be approximated by a map $g \in \mathbb{G}$ within any $\epsilon > 0$ neighborhood, i.e., $\rho(h, g) < \epsilon$.

Lemma 3. *Let (x_1, y_1) and (x_2, y_2) be two dyadic points where $x_1 < x_2$ and $y_1 < y_2$. Suppose that $y_2 - y_1 \geq x_2 - x_1$. If $\frac{y_2 - y_1}{x_2 - x_1} \neq 2^k$ for any integer k , then a dyadic point (x_3, y_3) exists with $x_1 < x_3 < x_2$, $y_1 < y_3 < y_2$ such that the slopes between $(x_1, y_1), (x_3, y_3)$ and between $(x_2, y_2), (x_3, y_3)$ are both in the form of 2^k for non-negative integer k .*

Proof. Let

$$\frac{y_3 - y_1}{x_3 - x_1} = 2^{k_1}, \frac{y_3 - y_2}{x_3 - x_2} = 2^{k_2}.$$

Then

$$x_3 = x_1 + \frac{2^{-k_2}(y_2 - y_1) - (x_2 - x_1)}{2^{k_1 - k_2} - 1}.$$

Two integer solutions are given by

$$\begin{cases} k_2 = \left\lfloor \log_2 \frac{y_2 - y_1}{x_2 - x_1} \right\rfloor \\ k_1 = k_2 + 1, \end{cases} \quad \text{and} \quad \begin{cases} k_2 = \left\lceil \log_2 \frac{y_2 - y_1}{x_2 - x_1} \right\rceil \\ k_1 = k_2 - 1, \end{cases}$$

It is easy to verify that in either solution, (x_3, y_3) is dyadic and $x_1 < x_3 < x_2$ and $y_1 < y_3 < y_2$. □

Point (x_3, y_3) in Lemma 3 is referred to as a *partition point* between points (x_1, y_1) and (x_2, y_2) .

Proposition 4. *For any increasing continuous map $a : [0, 1] \rightarrow [0, 1]$ and any $\epsilon > 0$, map $f \in \mathbb{F}$ exists such that $\rho(a, f) < \epsilon$.*

Proof. Because the set of dyadic points is dense and map a is increasing, a set of dyadic points (x_i, y_i) , for $i = 0, 1, \dots, n$, exist such that $x_0 = 0, x_n = 1, x_i < x_j$ and $y_i < y_j$ if $i < j$, and $a(x_i) - a(x_{i-1}) < \frac{\epsilon}{3}$ and $|y_i - a(x_i)| < \frac{\epsilon}{3}$ for all i . Connect point (x_{i-1}, y_{i-1}) and point (x_i, y_i) directly if the slope between them is in the form of 2^k for integer k or otherwise via a partition point between them defined in Lemma 3. The resultant map is $f \in \mathbb{F}$. For $x \in [x_{i-1}, x_i]$,

$$\begin{aligned} |a(x) - f(x)| &\leq \max(a(x_i), y_i) - \min(a(x_{i-1}), y_{i-1}) \\ &\leq |a(x_i) - y_i| + |a(x_{i-1}) - y_{i-1}| + |a(x_i) - a(x_{i-1})| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Hence, $\rho(a, f) < \epsilon$. □

Theorem 5. \mathbb{G} is dense in $C(\lambda)$. That is, for any $b \in C(\lambda)$ and $\epsilon > 0$, map $g \in \mathbb{G}$ exists such that $\rho(b, g) < \epsilon$.

Proof. Given $b \in C(\lambda)$ and $\epsilon > 0$, it has been shown in [8] that $h \in PA(\lambda)$ exists such that $\rho(b, h) < \epsilon/2$. In this proof, let $g = h$ initially and then perturb g in the following three steps such that $\rho(g_{\text{old}}, g_{\text{new}}) < \epsilon/6$ in each step, where g_{old} and g_{new} represent map g before and after each step of perturbation, respectively, and eventually make g an element of \mathbb{G} .

Let $\{(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)\}$ be the set of breakpoints of g , where $x_0 = 0, x_n = 1$. Points (x_0, y_0) and (x_n, y_n) are the two endpoints. Not all x_i, y_i are dyadic. The first step is to eliminate (x_i, y_i)

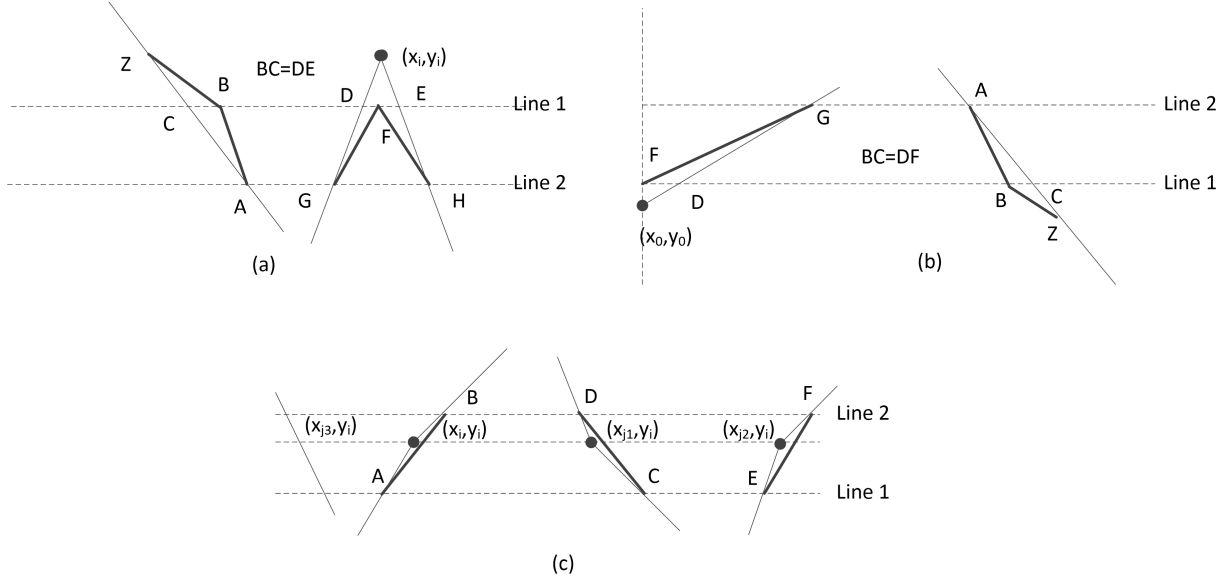


Figure 2: Step 1 of the proof of Theorem 5: eliminate any breakpoint whose y -coordinate is not dyadic. In the figure, the thin and thick segments represent the graph of g before and after the replacement respectively. In (a), type II breakpoint (x_i, y_i) is replaced by point F while segment ZA is replaced by ZB and BA to preserve λ . In (b), endpoint (x_0, y_0) is replaced by point F while segment ZA is replaced by ZB and BA . In (c), type I breakpoints $(x_i, y_i), (x_{j_1}, y_i), (x_{j_2}, y_i)$ are eliminated by directly connecting AB, CD, EF .

if y_i is not dyadic. The second step is to eliminate (x_i, y_i) if x_i is not dyadic. The third step is to eliminate segments whose slopes are not in the form of $\pm 2^k$ with integer k . Measure is preserved in each step.

Step 1. First, suppose that point (x_i, y_i) is a type II breakpoint and y_i is not dyadic. Figure 2(a) shows how (x_i, y_i) is eliminated. Because g is continuous and onto $[0, 1]$, there exists another point Z on g such that $Z_y = y_i$ and the left side and right side derivatives of Z are of the same sign. Point Z is not necessarily a breakpoint itself. Without loss of generality, suppose that the right side derivative of (x_i, y_i) is negative. Let line 2 be a horizontal line $y = \hat{y}_i$ where \hat{y}_i is dyadic, $0 < \hat{y}_i < y_i$ and $y_i - \hat{y}_i < \frac{\epsilon}{12}$. Choose \hat{y}_i sufficiently close to y_i such that no breakpoint exists whose y -coordinate falls in (\hat{y}_i, y_i) . Let line 1 be a horizontal line $y = \tilde{y}_i$ where \tilde{y}_i is dyadic, $\hat{y}_i < \tilde{y}_i < y_i$. Let D and E be the two points where the left and right side affine segments of g connecting (x_i, y_i) intersect line 1, and G and H be the two points where they intersect line 2. Let C and A be the two points where g connecting Z intersects lines 1 and 2 respectively. Choose \tilde{y}_i sufficiently close to y_i such that $|C_x - Z_x| + E_x - D_x < |A_x - Z_x|$. Such a \tilde{y}_i exists because $C_x - Z_x \rightarrow 0$ and $E_x - D_x \rightarrow 0$ as $\tilde{y}_i \rightarrow y_i$. Let F be any point on line 1 with $D_x < F_x < E_x$. Let point B be on line 1 where $BC = DE$ and $|Z_x - B_x| > |Z_x - C_x|$. Replace (x_i, y_i) with F by connecting G, F and H, F . Replace the portion of g between Z and A with segments ZB and BA . Because $BC = DE$, measure is preserved between the horizontal line $y = y_i$ and line 1 and between line 1 and line 2. Therefore, a type II breakpoint (x_i, y_i) is eliminated while six new breakpoints are added: A, B, F, G, H all have dyadic y -coordinate and Z is a type I breakpoint with non-dyadic y -coordinate. $\rho(g_{\text{old}}, g_{\text{new}}) < \frac{\epsilon}{12}$. Repeat the preceding procedure, one can eliminate all type II breakpoints with non-dyadic y -coordinates.

Endpoints (x_0, y_0) and (x_n, y_n) where y_0 or y_n is not dyadic can be eliminated analogously as shown Figure 2(b).

After the preceding procedure, the only remaining breakpoints that have non-dyadic y -coordinates are of type I. If (x_i, y_i) is one such breakpoint, then there exists at least another breakpoint (x_j, y_i) where

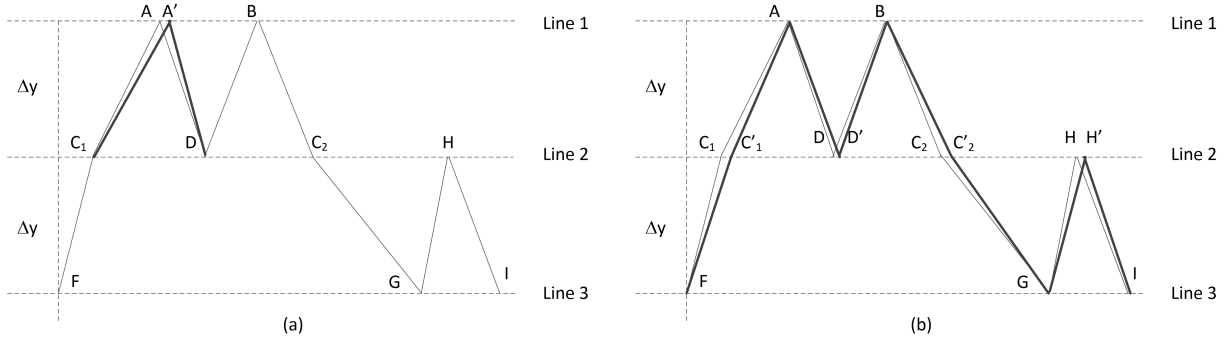


Figure 3: Step 2 of the proof of Theorem 5: eliminate any breakpoint whose x -coordinate is not dyadic. In the figure, the thin and thick segments represent the graph of g before and after the replacement respectively. In (a), point A , which is not dyadic, is replaced by dyadic point A' . In (b), C_1, D, C_2, H are not dyadic and are replaced by dyadic points C'_1, D', C'_2, H' respectively.

$x_j \neq x_i$. Figure 2(c) illustrates an example where two such breakpoints (x_{j_1}, y_i) and (x_{j_2}, y_i) exist. It is possible that points with y -coordinate equal to y_i exist and are not a breakpoint, such as (x_{j_3}, y_i) in the figure. Let line 1 and line 2 be horizontal lines $y = \hat{y}_i$ and $y = \tilde{y}_i$, respectively, where \hat{y}_i and \tilde{y}_i are both dyadic, $\hat{y}_i < y_i < \tilde{y}_i$ and $\tilde{y}_i - \hat{y}_i < \frac{\epsilon}{12}$. Let \hat{y}_i and \tilde{y}_i be sufficiently close to y_i that no breakpoint exists whose y -coordinate is not equal to y_i and falls in (\hat{y}_i, \tilde{y}_i) . Let A, C, E be the points where g intersects line 1 and B, D, F be the points where g intersects line 2. Replace the portion g between A and B with segment AB , between C and D with segment CD , and between E and F with segment EF . Measure is preserved between line 1 and line 2 after the replacement, because measure is preserved before the replacement between line 1 and the horizontal line $y = y_i$ and between line 2 and the horizontal line $y = y_i$. Therefore, the three type I breakpoints $(x_i, y_i), (x_{j_1}, y_i), (x_{j_2}, y_i)$ are eliminated while six new breakpoints are added: A, B, C, D, E, F all have dyadic y -coordinates. $\rho(g_{\text{old}}, g_{\text{new}}) < \frac{\epsilon}{12}$. Repeat the preceding procedure, one can eliminate all type I breakpoints with non-dyadic y -coordinates.

In step 1, $g(x)$ for any x is perturbed at most twice. For example, in Figure 2(a) x close to and greater than Z_x is perturbed once and will be perturbed again when type I breakpoint Z is to be eliminated as in Figure 2(c). On the other hand, point (x_i, y_i) is perturbed only once shown in Figure 2(a). Hence, at the end of step 1, $\rho(g, h) < \frac{\epsilon}{12} + \frac{\epsilon}{12} = \frac{\epsilon}{6}$.

Step 2. Eliminate (x_i, y_i) if x_i is not dyadic. Because g is piecewise affine on $[0, 1]$, g satisfies a Lipschitz condition, i.e., $|g(x_1) - g(x_2)| < K|x_1 - x_2|$ for some fixed number K and any $x_1, x_2 \in [0, 1]$. Consider a set of horizontal equally-spaced dyadic lines, $y = i \cdot 2^{-M}$ for $i = 0, 1, \dots, 2^M$, where the spacing between any adjacent dyadic lines is equal to $\Delta y = 2^{-M}$. Let $2^{-M} < \frac{\epsilon}{6}$ and is sufficiently small that all the breakpoints of g are on the lines. The lines are referred to as lines 1, 2, 3 and so on, as shown in Figure 3.

Because g is onto $[0, 1]$, one or multiple points of g exist on line 1. Figure 3(a) shows two such points A, B . If A_x is not dyadic, then replace it with sufficiently close dyadic number A'_x , i.e., with $|A_x - A'_x| < \frac{\epsilon}{6K}$. Because A_x is not dyadic, A cannot be an endpoint and thus must be a type II breakpoint and connect to two points of the graph of g on line 2, one to the left, C_1 , and one to the right, D . Replace the original affine segments C_1A, DA with C_1A', DA' . Because $D_x - A'_x + A'_x - C_{1,x} = D_x - A_x + A_x - C_{1,x}$, measure is preserved between line 1 and line 2 after the replacement. Therefore, all the points on line 1 are now dyadic.

Now consider the set of points of the graph of g on line 2 that are not dyadic. If a point is a type II breakpoint, e.g., D and H shown in Figure 3(b), it can be replaced like A on line 1. Otherwise, it must connect to one point of g on line 1 and another point of g on line 3, one to the left and one to the right.

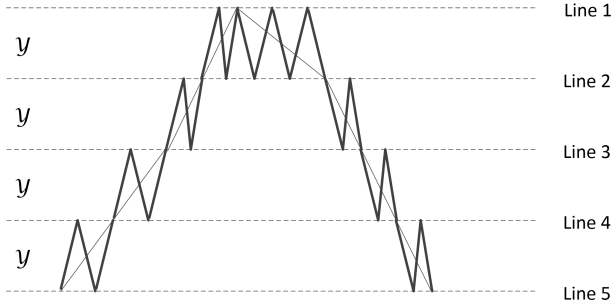


Figure 4: Step 3 of the proof of Theorem 5: connect breakpoints on adjacent horizontal dyadic lines. In the figure, the thin and thick segments represent the graph of g before and after the replacement respectively. The slopes of the thin segments are not in the form of $\pm 2^k$ for integer k . The slopes of the thick segments are in the form of $\pm 2^k$ for integer k .

Let C_1, C_2, \dots be these points. Figure 3(b) shows C_1, C_2 on line 2. Point C_1 connects to the right to A on line 1 and to the left to F on line 3. Point C_2 connects to the left to B on line 1 and to the right to G on line 3. Let $\text{sgn}_{C_i} = 1$ (or respectively, $\text{sgn}_{C_i} = -1$) if C_i connects to the left (or respectively, right) on line 1. In Figure 3(b), $\text{sgn}_{C_1} = -1$ and $\text{sgn}_{C_2} = 1$. If any $C_{i,x}$ are not dyadic, then replace them with sufficiently close dyadic numbers $C'_{i,x}$, i.e., with $|C_{i,x} - C'_{i,x}| < \frac{\epsilon}{6K}$, such that $\text{sgn}_{C'_i} = \text{sgn}_{C_i}$ and $\sum_i \text{sgn}_{C'_i} C'_{i,x} = \sum_i \text{sgn}_{C_i} C_{i,x}$. Such dyadic numbers $C'_{i,x}$ exist because of λ -preservation and because $\Delta y = 2^{-M}$ and all points on line 1 are dyadic, and therefore $\sum_i \text{sgn}_{C_i} C_{i,x}$ must be dyadic even though individual $C_{i,x}$ are not dyadic. Measure is thus preserved between line 1 and line 2 and between line 2 and line 3 after the replacement. Therefore, all the points on line 2 are now dyadic.

Repeat the same procedure for all the $2^M + 1$ lines. Therefore all the breakpoints are now dyadic. In step 2, a breakpoint is perturbed horizontally at most by $\frac{\epsilon}{6K}$. Hence, at the end of step 2, $\rho(g, h) < \frac{\epsilon}{6} + K \cdot \frac{\epsilon}{6K} = \frac{\epsilon}{3}$.

Step 3. The final step of the proof is to connect these breakpoints between adjacent horizontal dyadic lines with piecewise affine segments whose slopes are in the form of $\pm 2^k$ for integer k . Let \mathcal{Y} be the interval between two adjacent horizontal dyadic lines. Because no breakpoint exists on \mathcal{Y}° , $g^{-1}(\mathcal{Y})$ can be written as $\bigcup_i \mathcal{I}_i$, where $\{\mathcal{I}_i\}$ have mutually disjoint interiors, $g(\mathcal{I}_i) = \mathcal{Y}$ for all i , and g is affine on all \mathcal{I}_i . From step 2, the endpoints of any \mathcal{I}_i are dyadic. Let $|\mathcal{I}_i| = l_i \cdot 2^{-N_i}$ for integers l_i and N_i where l_i is an odd number. Because of λ -preservation, $|\mathcal{Y}| = \sum_i |\mathcal{I}_i|$, i.e., $2^{-M} = \sum_i l_i \cdot 2^{-N_i}$. Thus, $N_i \geq M$.

Replace the affine segment of g on \mathcal{I}_i with a window perturbation, as illustrated in Figure 4. The window perturbation is l_i -fold where each leg has slope of $\pm 2^{N_i - M}$ and covers the same interval as the original affine segment. Therefore measure is preserved after the replacement.

In step 3, the maximum vertical perturbation cannot exceed the spacing between any adjacent dyadic lines, which is $2^{-M} < \frac{\epsilon}{6}$.

Hence, at the end of step 3, the resultant g is an element of \mathbb{G} and $\rho(h, g) < \frac{\epsilon}{3} + \frac{\epsilon}{6} = \frac{\epsilon}{2}$. $\rho(b, g) \leq \rho(b, h) + \rho(h, g) < \epsilon$. This completes the proof. \square

4 Mixing

Topological transitivity and mixing properties of dynamical systems are widely studied in the literature. There are several different versions of such properties, including topological transitivity (TT), strong transitivity (ST), exact transitivity (ET), weak mixing (WM), topological mixing (TM), and locally

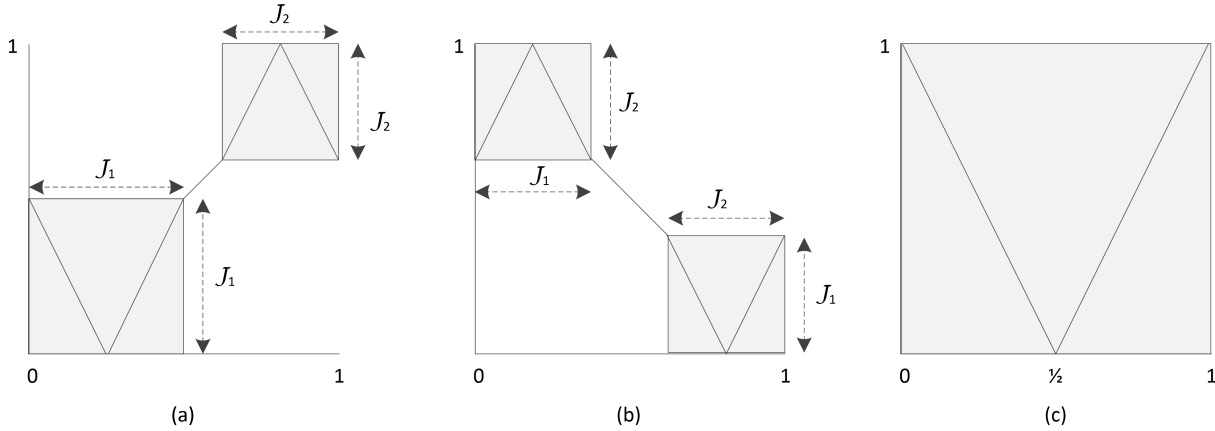


Figure 5: Examples of $g \in \mathbb{G}$. Map g is not TM or LEO in (a) and (b) and is TM and LEO in (c). In (a) $g(\mathcal{J}_1) = \mathcal{J}_1$ and $g(\mathcal{J}_2) = \mathcal{J}_2$. In (b) $g(\mathcal{J}_1) = \mathcal{J}_2$ and $g(\mathcal{J}_2) = \mathcal{J}_1$.

eventually onto (LEO). It is shown in [9] that many of these versions, such as TT, ET, WM, and TM, are implied by LEO but not the other way around. Thus, LEO is the strongest version among them. This section focuses on LEO and TM.

Definition 7 (Topological Mixing (TM)). An interval map h is TM if for all nonempty open sets U, V in $[0, 1]$, there exists an integer $N \geq 0$ such that $\forall n \geq N, f^n(U) \cap V \neq \emptyset$.

Definition 8 (Locally Eventually Onto (LEO)). An interval map h is LEO if for every nonempty open set U in $[0, 1]$ there is an integer N such that $h^N(U) = [0, 1]$.

Remark. It can be shown ([1, Proposition. 2.8]) that h is TM if and only if $\forall \epsilon > 0$ and open $U \subset [0, 1]$, there is an integer N such that $h^n(U) \supset [\epsilon, 1 - \epsilon]$ for any $n \geq N$. However, it is not necessary that $h^n(U) = [0, 1]$. Clearly, LEO implies TM. The difference between LEO and TM lies at two endpoints.

Remark. If h is LEO or TM, then its trajectory is sensitive to initial conditions in the sense that two arbitrarily close initial conditions x_1 and x_2 lead to divergent trajectories as $h^n(x_1)$ and $h^n(x_2)$ eventually spread over the entire interval $(0, 1)$ for $n \geq 0$.

Lemma 6 (Barge and Martin, 1985, [10]). *If a continuous interval map h has a dense set of periodic points, then a collection of intervals $\{\mathcal{J}_1, \mathcal{J}_2, \dots\}$ of $[0, 1]$ exist with mutually disjoint interiors such that for each $i, h^2(\mathcal{J}_i) = \mathcal{J}_i, h(\mathcal{J}_i) = \mathcal{J}_j$ for some $j \geq 1, h^{-1}(h(\mathcal{J}_i)) = \mathcal{J}_i$, and $h^2(x) = x$ on $[0, 1] \setminus \bigcup_{i \geq 1} \mathcal{J}_i^\circ$. If $|\{\mathcal{J}_1, \mathcal{J}_2, \dots\}| > 1$, then there can only be two cases. In the first case, $h(\mathcal{J}_i) = \mathcal{J}_i$ for all i and $h(x) = x$ on $[0, 1] \setminus \bigcup_{i \geq 1} \mathcal{J}_i^\circ$. In the second case, $h(\mathcal{J}_i) > h(\mathcal{J}_j), \forall \mathcal{J}_i < \mathcal{J}_j, h(\mathcal{J}_i) = \mathcal{J}_i$ for at most one i and $h(x) = 1 - x$ on $[0, 1] \setminus \bigcup_{i \geq 1} \mathcal{J}_i^\circ$.*

Recall from Section 2.4 that for each map $g \in \mathbb{G}$, the set of periodic points is dense on $[0, 1]$. A collection of $\{\mathcal{J}_1, \mathcal{J}_2, \dots\}$ of g exist to have the properties stated in Lemma 6.

The two cases of Lemma 6 are illustrated in Figure 5(a) and (b). Set $\{\mathcal{J}_1, \mathcal{J}_2, \dots\}$ is not unique when $[0, 1] \setminus \bigcup_{i \geq 1} \mathcal{J}_i^\circ \neq \emptyset$. For example, in Figure 5(a) g is an affine segment with slope 1 on interval $[\mathcal{J}_1^1, \mathcal{J}_2^0]$. Let $\mathcal{J}'_1 = [\mathcal{J}_1^0, \mathcal{J}_1^1 + \delta_1]$ and $\mathcal{J}'_2 = [\mathcal{J}_2^0 - \delta_2, \mathcal{J}_2^1]$ with $\delta_1, \delta_2 > 0$ and $\delta_1 + \delta_2 \leq \mathcal{J}_2^0 - \mathcal{J}_1^1$. In Figure 5(b) g is an affine segment with slope -1 on interval $[\mathcal{J}_1^1, \mathcal{J}_2^0]$. Let $\mathcal{J}'_1 = [\mathcal{J}_1^0, \mathcal{J}_1^1 + \delta]$ and $\mathcal{J}'_2 = [\mathcal{J}_2^0 - \delta, \mathcal{J}_2^1]$ with $0 < \delta \leq \frac{\mathcal{J}_2^0 - \mathcal{J}_1^1}{2}$. In either case, $\{\mathcal{J}'_1, \mathcal{J}'_2\}$ has the same properties as $\{\mathcal{J}_1, \mathcal{J}_2\}$ as far as Lemma 6 is concerned. Intervals $\mathcal{J}'_1, \mathcal{J}'_2$ can shrink in length to become $\mathcal{J}_1, \mathcal{J}_2$. More precisely, a collection of intervals $\{\mathcal{J}_1, \mathcal{J}_2, \dots\}$ in Lemma 6 is said to be of minimum length if there does not exist a distinct collection $\{\mathcal{J}'_1, \mathcal{J}'_2, \dots\}$, also satisfying Lemma 6, such that $\mathcal{J}'_i \subseteq \mathcal{J}_i$ for $i = 1, 2, \dots$.

Lemma 7. *Let $g \in \mathbb{G}$. Suppose that $\{\mathcal{J}_1, \mathcal{J}_2, \dots\}$ is a collection of g in Lemma 6. If $\{\mathcal{J}_1, \mathcal{J}_2, \dots\}$ is of minimum length, then the endpoints of each \mathcal{J}_i are dyadic.*

Proof. First suppose that $|\{\mathcal{J}_1, \mathcal{J}_2, \dots\}| = 1$. That is, $\{\mathcal{J}_1, \mathcal{J}_2, \dots\} = \{\mathcal{J}_1\}$. If $\mathcal{J}_1 = [0, 1]$, then the proof is already done. If $\mathcal{J}_1^0 > 0$ and $\mathcal{J}_1^1 = 1$, then by Lemma 6 $g(x) = x$ on $[0, \mathcal{J}_1^0]$. A type II breakpoint (x_0, y_0) must exist such that $y_0 = \mathcal{J}_1^0$, for otherwise $\delta > 0$ exists such that the set $g^{-1}(y)$ consists of only one element for $y \in [\mathcal{J}_1^0, \mathcal{J}_1^0 + \delta]$ and because of λ -preservation, $g(x) = x$ on $[\mathcal{J}_1^0, \mathcal{J}_1^0 + \delta]$. Thus, \mathcal{J}_1 can shrink to $[\mathcal{J}_1^0 + \delta, \mathcal{J}_1^1]$. Therefore, \mathcal{J}_1^0 is dyadic by Lemma 1. Similarly, if $\mathcal{J}_1^0 = 0$ and $\mathcal{J}_1^1 < 1$, then \mathcal{J}_1^1 can be shown to be dyadic. Now suppose $\mathcal{J}_1^0 > 0$ and $\mathcal{J}_1^1 < 1$. By Lemma 6 there can only be two cases. In the first case, $g(x) = x$ on $[0, 1] \setminus \mathcal{J}_1^0$. Type II breakpoints (x_0, y_0) and (x_1, y_1) must exist such that $y_0 = \mathcal{J}_1^0$ and $y_1 = \mathcal{J}_1^1$. Therefore, \mathcal{J}_1^0 and \mathcal{J}_1^1 are both dyadic by Lemma 1. In the second case, $g(x) = 1 - x$ on $[0, 1] \setminus \mathcal{J}_1^0$. A type II breakpoint (x_0, y_0) must exist such that $y_0 = \mathcal{J}_1^0$ or $y_0 = \mathcal{J}_1^1$, for otherwise $\delta > 0$ exists such that the set $g^{-1}(y)$ consists of only one element for $y \in [\mathcal{J}_1^0, \mathcal{J}_1^0 + \delta]$ and $y \in [\mathcal{J}_1^1 - \delta, \mathcal{J}_1^1]$ and thus \mathcal{J}_1 can shrink to $[\mathcal{J}_1^0 + \delta, \mathcal{J}_1^1 - \delta]$. Therefore, at least one of \mathcal{J}_1^0 and \mathcal{J}_1^1 are dyadic. By Lemma 1, both of them are dyadic because $g(\mathcal{J}_1^0) = \mathcal{J}_1^1$ and $g(\mathcal{J}_1^1) = \mathcal{J}_1^0$.

Next suppose that $|\{\mathcal{J}_1, \mathcal{J}_2, \dots\}| > 1$. Consider the two cases of Lemma 6. In the first case, $g(x) = x$ on $[0, 1] \setminus \bigcup_{i \geq 1} \mathcal{J}_i^0$. Type II breakpoints (x_0, y_0) and (x_1, y_1) must exist such that $y_0 = \mathcal{J}_i^0$ and $y_1 = \mathcal{J}_i^1$ for each i , for otherwise \mathcal{J}_i can shrink similar to what is shown above. Numbers \mathcal{J}_i^0 and \mathcal{J}_i^1 are both dyadic. In the second case, $g(x) = 1 - x$ on $[0, 1] \setminus \bigcup_{i \geq 1} \mathcal{J}_i^0$. For each i , $h(\mathcal{J}_i) = \mathcal{J}_j$. If $j = i$, type II breakpoint (x_0, y_0) must exist such that $y_0 = \mathcal{J}_i^0$ or $y_0 = \mathcal{J}_i^1$, for otherwise \mathcal{J}_i can shrink similar to what is shown above. Numbers \mathcal{J}_i^0 and \mathcal{J}_i^1 are both dyadic. Now consider $j \neq i$. Without loss of generality, suppose that $\mathcal{J}_j < \mathcal{J}_i$. Thus, $g(\mathcal{J}_i^0) = \mathcal{J}_j^1$. A type II breakpoints (x_0, y_0) must exist such that $y_0 = \mathcal{J}_i^0$ or $y_0 = \mathcal{J}_j^1$, for otherwise $\delta > 0$ exists such that the set $g^{-1}(y)$ consists of only one element for $y \in [\mathcal{J}_i^0, \mathcal{J}_i^0 + \delta]$ and $y \in [\mathcal{J}_j^1 - \delta, \mathcal{J}_j^1]$ and thus \mathcal{J}_i can shrink to $[\mathcal{J}_i^0 + \delta, \mathcal{J}_i^1]$ and \mathcal{J}_j can shrink to $[\mathcal{J}_j^0, \mathcal{J}_j^1 - \delta]$. Thus, at least one of \mathcal{J}_i^0 and \mathcal{J}_j^1 are dyadic. By Lemma 1, both of them are dyadic because $g(\mathcal{J}_i^0) = \mathcal{J}_j^1$. Analogously, both \mathcal{J}_i^1 and \mathcal{J}_j^0 can be shown to be dyadic. \square

Lemma 8 (Bobok and Troubetzkoy, 2019, [3, Lemma. 5]). *In Lemma 6, h is TM if and only if the collection of intervals satisfy $\{\mathcal{J}_1, \mathcal{J}_2, \dots\} = \{[0, 1]\}$, and h is LEO if and only if in addition both of the sets $h^{-2}(0) \cap (0, 1)$ and $h^{-2}(1) \cap (0, 1)$ are non-empty.*

Figure 5 provides three examples of $g \in \mathbb{G}$, two of which are not TM or LEO and the third one is LEO. In (a) and (b), $\mathcal{J}_1 \cup \mathcal{J}_2 \subset [0, 1]$ with $\mathcal{J}_1^0 \cap \mathcal{J}_2^0 = \emptyset$ and $g^2(\mathcal{J}_i) = (\mathcal{J}_i)$ for $i = 1, 2$. As a result, if $U \subset \mathcal{J}_i$, then given n , $g^n(U) \subset \mathcal{J}_j$ for either $j = 1$ or $j = 2$ but it is impossible that $g^n(U) = [0, 1]$. Therefore, g in (a) and (b) is not TM or LEO. In (c), such a partition of separate \mathcal{J}_1 and \mathcal{J}_2 does not exist and $\{\mathcal{J}_1, \mathcal{J}_2, \dots\} = \{[0, 1]\}$. Thus, g in (c) is TM by Lemma 8. In addition, because $\frac{1}{2} \in g^{-2}(0)$ and $\frac{1}{2} \in g^{-2}(1)$, g in (c) is LEO.

As remarked, in general, LEO implies TM and the converse does not hold. However, the two are equivalent for $g \in \mathbb{G}$ as stated in the following theorem.

Theorem 9. *If $g \in \mathbb{G}$ is TM, then g is LEO.*

Proof. We will prove the theorem by contradiction. Assume that $g^{-2}(0) \cap (0, 1) = \emptyset$. First we will show by contradiction that $g^{-1}(0) \cap (0, 1) = \emptyset$. Assume that $c \in g^{-1}(0)$ and $0 < c < 1$. Because $g^{-1}(c) \subseteq g^{-2}(0)$, $g^{-1}(c) \cap (0, 1) = \emptyset$, which is impossible because g is continuous and onto $[0, 1]$. Therefore, $g^{-1}(0) \subseteq \{0, 1\}$.

If $g^{-1}(0) = \{0\}$, then $g(x)$ is an affine segment with slope 1 on $[0, \delta]$ for some sufficiently small $\delta > 0$. Thus $g(x)$ is not TM because $g^n((0, \delta)) = (0, \delta)$ for any n and does not mix with $(\delta, 1)$. Contradiction with the hypothesis.

If $g^{-1}(0) = \{1\}$, then $g(x)$ is an affine segment with slope -1 on $[1 - \delta_1, 1]$ for some sufficiently small $\delta_1 > 0$. Moreover, for $g^{-2}(0) \cap (0, 1) = \emptyset$, it follows that $g^{-1}(1) = \{0\}$. The graph of $g(x)$ is an affine segment with slope -1 on $[0, \delta_2]$ for some sufficiently small $\delta_2 > 0$. Let $\delta = \min(\delta_1, \delta_2)$. $g^n((0, \delta)) = (0, \delta)$ for even n and $g^n((0, \delta)) = (1 - \delta, 1)$ for odd n . Thus $g(x)$ is not TM because $g^n((0, \delta))$ for any n and

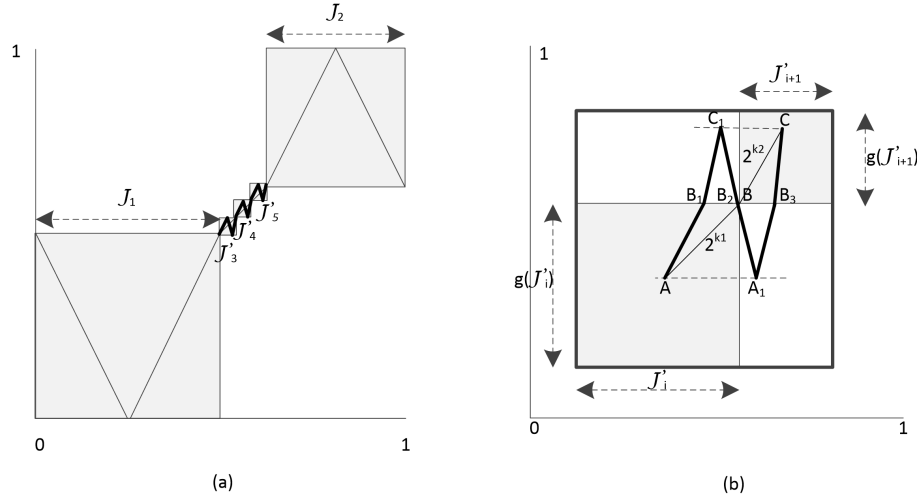


Figure 6: Perturbation in the proof of Theorem 10. (a) Add new $\mathcal{J}'_3, \mathcal{J}'_4, \mathcal{J}'_5$ such that $[0, 1] = \bigcup_{i \geq 1} \mathcal{J}'_i$. (b) Merge \mathcal{J}_i and \mathcal{J}_{i+1} into one interval by replacing the thin segments AB, BC with thick segments $AB_1, B_1C_1, C_1B_2, B_2A_1, A_1B_1, B_1C$.

does not mix with $(\delta, 1 - \delta)$. Contradiction with the hypothesis.

If $g^{-1}(0) = \{0, 1\}$, then $g^{-1}(1) \subset (0, 1)$. Therefore, $g^{-2}(0) \cap (0, 1) \neq \emptyset$. Contradiction with the assumption.

Hence, $g^{-2}(0) \cap (0, 1) \neq \emptyset$. We can analogously show that $g^{-2}(1) \cap (0, 1) \neq \emptyset$. By Lemma 8, g is LEO. \square

Theorem 10. Denote by \mathbb{G}_{LEO} the subset of \mathbb{G} whose elements are LEO. \mathbb{G}_{LEO} is dense in $C(\lambda)$.

Proof. The idea is to further perturb g obtained in the proof of Theorem 5 to meet the conditions required in Lemma 8, thereby making g LEO. Specifically, map g obtained in the proof of Theorem 5 is an element of \mathbb{G} , and thus the set of periodic points of g is dense on $[0, 1]$ from Section 2.4. A collection of intervals $\{\mathcal{J}_1, \mathcal{J}_2, \dots\}$ of $[0, 1]$ exist to have the properties stated in Lemma 6.

Step 1. Add new intervals to the set $\{\mathcal{J}_1, \mathcal{J}_2, \dots\}$ such that the new set $\{\mathcal{J}'_1, \mathcal{J}'_2, \dots\}$ covers $[0, 1]$ in g_{new} as illustrated in Figure 6(a). By Lemma 6, if $x \in [0, 1] \setminus \bigcup_{i \geq 1} \mathcal{J}'_i$, then the derivative of g is either 1 or -1 . Divide $[0, 1] \setminus \bigcup_{i \geq 1} \mathcal{J}'_i$ into a number of intervals with mutually disjoint interiors and each has an maximum length smaller than $\frac{\epsilon}{2}$ and dyadic endpoints. Such a division exists because the endpoints of $\{\mathcal{J}_i\}$ are all dyadic by Lemma 7. If the affine segment on one of the intervals has the slope 1, then replace it with a 3-fold window perturbation of slopes $2^1, -2^2, 2^2$ on the three legs respectively; otherwise, the affine segment has the slope -1 and replace it with a 3-fold window perturbation of slopes $-2^1, 2^2, -2^2$ on the three legs respectively.

After step 1, $g_{\text{new}} \in \mathbb{G}$ and $\rho(g_{\text{old}}, g_{\text{new}}) < \frac{\epsilon}{2}$. Combine the original $\mathcal{J}_1, \mathcal{J}_2, \dots$ and the newly added intervals to become $\{\mathcal{J}'_1, \mathcal{J}'_2, \dots\}$. $\bigcup_{i \geq 1} \mathcal{J}'_i = [0, 1]$.

Step 2. Merge adjacent disjoint intervals. Consider adjacent disjoint intervals \mathcal{J}'_i and \mathcal{J}'_{i+1} . Let B be on the graph of g at the boundary point between \mathcal{J}'_i and \mathcal{J}'_{i+1} . The graph of g is an affine segment in a sufficiently small left and right neighborhood of B and the left and the right derivatives are of the same sign by Lemma 6. Suppose that the derivatives are both positive as shown in Figure 6(b). (The case of the derivatives being both negative can be proven analogously.) Let the left derivative be 2^{k_1} and the right derivative be 2^{k_2} . Let AB be the affine segment of g in \mathcal{J}'_i and BC be the affine segment of g in \mathcal{J}'_{i+1}

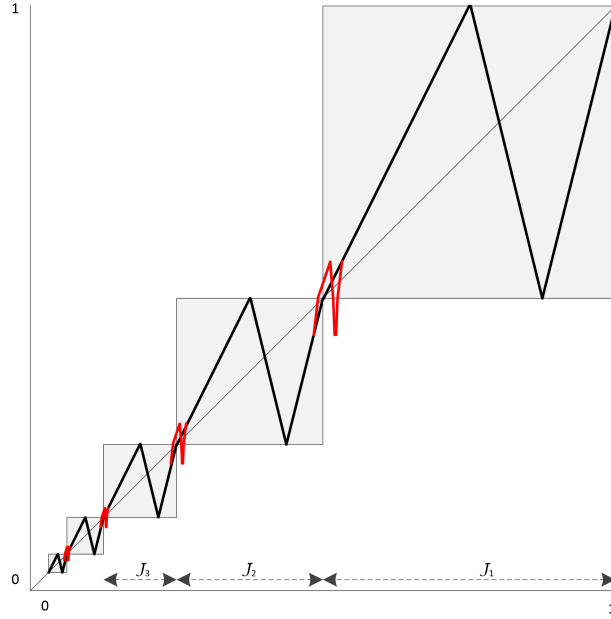


Figure 7: An example of h that is TM but is not LEO. First, replace the thin segment $h(x) = x$ with the black thick segments, which are the window perturbations on intervals $\mathcal{J}_1, \mathcal{J}_2, \dots$. Then, replace the black thick segments with the red thick segments at the boundaries between \mathcal{J}_i and \mathcal{J}_{i+1} for $i = 1, 2, \dots$ to be TM.

where $B_y - A_y = C_y - B_y = 2^{-M}$ for a large positive integer M such that $2^{-M} < \min\left(\frac{\epsilon}{4}, \frac{|\mathcal{J}'_i|}{2}, \frac{|\mathcal{J}'_{i+1}|}{2}\right)$ and no breakpoint exists on (A_x, B_x) or (B_x, C_x) . $B_x - A_x = (B_y - A_y) \cdot 2^{-k_1} = 2^{-M-k_1}$, $C_x - B_x = (C_y - B_y) \cdot 2^{-k_1} = 2^{-M-k_2}$. Thus A and C are both dyadic, because B is dyadic.

Replace segments AB and BC by the following six affine segments to merge \mathcal{J}'_i and \mathcal{J}'_{i+1} : AB_1 , B_1C_1 , C_1B_2 , B_2A_1 , A_1B_3 and B_3C . The connecting points A_1, C_1, B_1, B_2, B_3 are defined as follows: $C_{1,y} = C_y, A_{1,y} = A_y, B_{1,y} = B_{2,y} = B_{3,y} = B_y$ and $B_{1,x} - A_x = 2^{-M-k_1-1}, C_{1,x} - B_{1,x} = 2^{-M-k_2-1}, B_{2,x} - C_{1,x} = 2^{-M-k_2-2}, A_{1,x} - B_{2,x} = 2^{-M-k_1-2}, B_{3,x} - A_{1,x} = 2^{-M-k_1-2}, C_x - B_{3,x} = 2^{-M-k_2-2}$. It is easy to verify that g is still λ -preserving, the absolute values of the slopes of the affine segments are $2^{k_1+1}, 2^{k_1+2}, 2^{k_2+1}, 2^{k_2+2}$, and the newly added breakpoints $A, A_1, B_1, B_2, B_3, C, C_1$ are all dyadic. Therefore, $g_{\text{new}} \in \mathbb{G}$.

Repeat the preceding procedure for all i . The choice of M ensures that the perturbation done for all i does not overlap and $\rho(g_{\text{old}}, g_{\text{new}}) < \frac{\epsilon}{2}$ in step 2. After step 2, $\{\mathcal{J}'_1, \mathcal{J}'_2, \dots\}$ are all merged into $\{[0, 1]\}$.

Hence, after the preceding two steps of perturbation, $\rho(g_{\text{old}}, g_{\text{new}}) < \epsilon$. By Lemma 8, g is TM, and by Theorem 9, g is LEO. \square

Figure 7 shows an example of h that is TM but is not LEO. Specifically, partition $[0, 1]$ into countably infinitely many subintervals $\{\mathcal{J}_i\}$, where $\mathcal{J}_i = [2^{-i}, 2^{-i+1}]$ for $i = 1, 2, \dots$. Start with $h(x) = x$. Next replace $h(x)$ on \mathcal{J}_i with a 3-fold window perturbation, shown as the thick black segments. Then merge \mathcal{J}_i and \mathcal{J}_{i+1} as in step 2 of the proof of Theorem 10 illustrated in Figure 6(b). The merge is shown as the thick red segments in Figure 7. Let $h(0) = 0$. $h^{-1}(0) = \{0\}$. The endpoint $x = 0$ is not accessible and therefore h is not LEO. Recall that endpoint $x = 0$ is accessible if there exists $x \in (0, 1)$ such that $h^n(x) = 0$ for some $n > 0$. Note that this map h is not an element of \mathbb{G} , because there are infinitely many points at which h is not differentiable, although h meets all the other conditions of \mathbb{G} . Therefore this counterexample does not contradict Theorem 9.

5 Periodicity

Theorem 11 states a salient feature of \mathbb{G} .

Theorem 11. *Let $g \in \mathbb{G}$ and c be a dyadic number. Then point $(c, g(c))$ is preperiodic under the diagonal action $(x, y) \rightarrow (g(x), g(y))$.*

Proof. Let $0 = x_0 < \dots < x_n = 1$ be all the breakpoints of g . Let $x_i = \frac{l_i}{2^M}$ for $i = 0, 2, \dots, n$, $c = \frac{p_0}{2^M}$ and $g(c) = \frac{p_1}{2^M}$ for integers M and l_i, p_0, p_1 . Integers l_i, p_0, p_1 are not necessarily odd.

Let $\{x_{b_1}, x_{b_2}, \dots, x_{b_m}\}$ be the subset of breakpoints between c and $g(c)$, inclusive, where $m \geq 0$. Between $x = c$ and $x = g(c)$ there are $m + 1$ affine segments, each with a horizontal length in the form of $\frac{l}{2^M}$ and a slope in the form of $\pm 2^k$ for some integers l, k . The vertical displacement of any affine segment is in the form of $\pm \frac{l \cdot 2^k}{2^M}$. The sum of the vertical displacements of these $m + 1$ affine segments, equal to $g(g(c)) - g(c)$, is in the form of $\frac{l}{2^M}$ for some integer l . Given that $g(c) = \frac{p_1}{2^M}$, it follows that $g(g(c)) = \frac{p_2}{2^M}$ for some integer p_2 .

Repeating the preceding argument, it follows that $g^i(c) = \frac{p_i}{2^M}$ for integer p_i for all $i = 0, 1, 2, \dots$, with $0 \leq p_i \leq 2^M$. Because M is a finite number, the total number of distinct $\frac{p_i}{2^M}$ in $[0, 1]$ is finite. Hence, $g^{i_1}(c) = g^{i_2}(c)$ for some $i_1 \neq i_2$, and $(c, g(c))$ is a preperiodic point. \square

Definition 9 (Markov Map). A piecewise affine interval map is a Markov map if all breakpoints are preperiodic.

By definition, any breakpoint of $g \in \mathbb{G}$ is dyadic and thus preperiodic by Theorem 11. The following corollary follows immediately.

Corollary 12. *Any $g \in \mathbb{G}$ is a Markov map.*

By Theorem 10 and Corollary 12, \mathbb{G} that is both LEO and Markov is dense in $C(\lambda)$. Because \mathbb{G} is a subset of $PA(\lambda)$, this result is stronger than [3, Proposition. 7], which shows that $PA(\lambda)$ that are both LEO and Markov is dense in $C(\lambda)$. Corollary 12 provides an essential basis of the study of topological conjugacy in Section 9.

Definition 10 (Period of a Point). Suppose that x is a periodic point. The period of x is the least positive integer p such that $h^p(x) = x$.

Definition 11 (Chaotic Function). A map h is called chaotic if there exists a point x of period k for any positive integer k .

Li-Yorke theorem [11] states that if a periodic point x of period 3 exists, then h is chaotic. Periodic points of period 3 are thus of particular importance. The remainder of this section is to characterize the periods of periodic points of $g \in \mathbb{G}$.

Theorem 13. *Consider a continuous map h from $[0, 1]$ onto itself. If intervals $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \subseteq [0, 1]$ exist such that $\mathcal{I}_1 \subset \mathcal{I}_0, \mathcal{I}_2 \subset \mathcal{I}_0, \mathcal{I}_1^\circ \cap \mathcal{I}_2^\circ = \emptyset$, and $h(\mathcal{I}_1) = h(\mathcal{I}_2) = \mathcal{I}_0$, then a periodic point $x_0 \in \mathcal{I}_0$ of period 3 exists.*

Proof. Because $\mathcal{I}_1 \subset \mathcal{I}_0 = h(\mathcal{I}_2)$, an interval $\mathcal{I}_3 \subset \mathcal{I}_2$ exists such that $\mathcal{I}_1 = h(\mathcal{I}_3)$. Because $\mathcal{I}_3 \subset h(\mathcal{I}_2)$, an interval $\mathcal{I}_4 \subset \mathcal{I}_2$ exists such that $\mathcal{I}_3 = h(\mathcal{I}_4)$. Therefore, $\mathcal{I}_0 = h^3(\mathcal{I}_4)$. Because $\mathcal{I}_4 \subset \mathcal{I}_0$, by the Intermediate Value Theorem, $x_0 \in \mathcal{I}_4$ exists such that $h^3(x_0) = x_0$. Specifically, $x_1 \in \mathcal{I}_3$ and $x_2 \in \mathcal{I}_1$ exist such that $x_1 = h(x_0), x_2 = h(x_1)$ and $x_0 = h(x_2)$. Moreover, $\mathcal{I}_3 \cap \mathcal{I}_1 = \emptyset$ because $\mathcal{I}_3 \subset \mathcal{I}_2$ and $\mathcal{I}_1^\circ \cap \mathcal{I}_2^\circ = \emptyset$. Then, it follows that $\mathcal{I}_3 \cap \mathcal{I}_4 = \emptyset$ because $\mathcal{I}_1 = h(\mathcal{I}_3)$ and $\mathcal{I}_3 = h(\mathcal{I}_4)$. Thus x_0, x_1, x_2 are all distinct. Hence, the period of x_0 is 3. The proof is illustrated in Figure 8. \square

Corollary 14. *Consider a λ -preserving continuous map h from $[0, 1]$ onto itself. Suppose that an interval \mathcal{J} exists such that $h(\mathcal{J}) = \mathcal{J}$ and $h^{-1}(\mathcal{J}) = \mathcal{J}$. Let c be an endpoint of \mathcal{J} . If $h(c) = \mathcal{J}^0$ or $h(c) = \mathcal{J}^1$, and if $d_0 \in \mathcal{J}$ with $d_0 \neq c$ exists such that $h(d_0) = h(c)$, then a periodic point $x_0 \in \mathcal{J}$ of period 3 exists.*

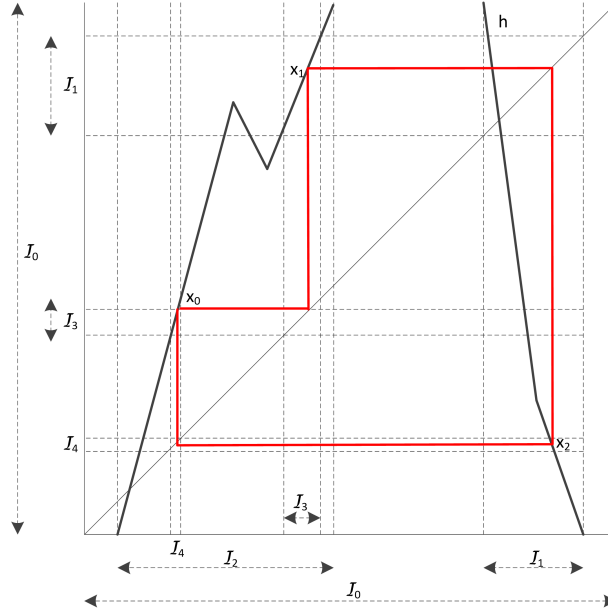


Figure 8: Proof of Theorem 13. The thick black lines represent h and the thick red lines show the period-3 trajectory of $x_0 \xrightarrow{h} x_1 \xrightarrow{h} x_2 \xrightarrow{h} x_0$.

Proof. Without loss of generality, suppose that $c = \mathcal{J}^0$. Suppose that $h(c) = \mathcal{J}^0$. The case of $h(c) = \mathcal{J}^1$ can be proved analogously.

From the hypothesis, there exists $d_1 \in \mathcal{J}$ such that $h(d_1) = \mathcal{J}^1$. If $d_0 > d_1$, then let $\mathcal{I}_1 = [c, d_1]$, $\mathcal{I}_2 = [d_1, d_0]$ and $\mathcal{I}_0 = \mathcal{J}$. The conclusion follows from Theorem 13. Otherwise, $d_0 < d_1$. Note that $h([c, d_0]) \supseteq [c, d_0]$, because otherwise $h^{-1}([d_0, \mathcal{J}^1]) \subset [d_0, \mathcal{J}^1]$ and λ is not preserved. Thus, there exist c' and d'_0 with $c < c' \leq d'_0 < d_0$ such that $h([c, c']) = h([d'_0, d_0]) = [c, d_0]$. Let $\mathcal{I}_1 = [c, c']$, $\mathcal{I}_2 = [d'_0, d_0]$ and $\mathcal{I}_0 = [c, d_0]$. The conclusion follows from Theorem 13. \square

Let $g \in \mathbb{G}$. Suppose that a collection of $\{\mathcal{J}_1, \mathcal{J}_2, \dots\}$ of g exist to have the properties stated in Lemma 6.

Case 1. Suppose that either $[0, 1] \setminus \bigcup_{i \geq 1} \mathcal{J}_i^\circ \neq \emptyset$ and $g(x) = x$ on $[0, 1] \setminus \bigcup_{i \geq 1} \mathcal{J}_i^\circ$, or $|\{\mathcal{J}_1, \mathcal{J}_2, \dots\}| > 1$ and $g(\mathcal{J}_i) = \mathcal{J}_i$ for all i . Then an interval $\mathcal{J} \subseteq \mathcal{J}_i$ for any i exists such that the hypothesis of Corollary 14 holds with $h(c) = c$. Hence, a periodic point of period 3 exists.

Case 2. Suppose that either $[0, 1] \setminus \bigcup_{i \geq 1} \mathcal{J}_i^\circ \neq \emptyset$ and $g(x) = 1 - x$ on $[0, 1] \setminus \bigcup_{i \geq 1} \mathcal{J}_i^\circ$, or $|\{\mathcal{J}_1, \mathcal{J}_2, \dots\}| > 1$ and $g(\mathcal{J}_i) > g(\mathcal{J}_j), \forall \mathcal{J}_i < \mathcal{J}_j$. If $g(\mathcal{J}_i) \neq \mathcal{J}_i, \forall i$, then the period of any periodic point is even, because $g^2(\mathcal{J}_i) = \mathcal{J}_i$. Otherwise, $\exists i_0$ such that $g(\mathcal{J}_{i_0}) = \mathcal{J}_{i_0}$. In this case, the period of any periodic point x is even when $x \in \mathcal{J}_i$ with $i \neq i_0$, and an interval $\mathcal{J} \subseteq \mathcal{J}_{i_0}$ exists such that the hypothesis of Corollary 14 holds with $h(c) = 1 - c$. Hence, a periodic point of period 3 exists.

Case 3. Suppose that $\{\mathcal{J}_1, \mathcal{J}_2, \dots\} = \{[0, 1]\}$. This case is of particular importance because of Lemma 8. Two results have been shown for an interval map h in the literature.

- First [1, Proposition. 2.18], if h is transitive, then h is TM if and only if it has a periodic point of odd period greater than 1.
- Second, define Sharkovsky's order of positive integers by

$$3 \triangleleft 5 \triangleleft 7 \triangleleft \dots \triangleleft 2 \cdot 3 \triangleleft 2 \cdot 5 \triangleleft 2 \cdot 7 \triangleleft \dots \triangleleft 2^2 \cdot 3 \triangleleft 2^2 \cdot 5 \triangleleft 2^2 \cdot 7 \triangleleft \dots \triangleleft 2^3 \triangleleft 2^2 \triangleleft 2 \triangleleft 1.$$

Sharkovsky's theorem [12] states that if h has a periodic point of period n , then h has periodic

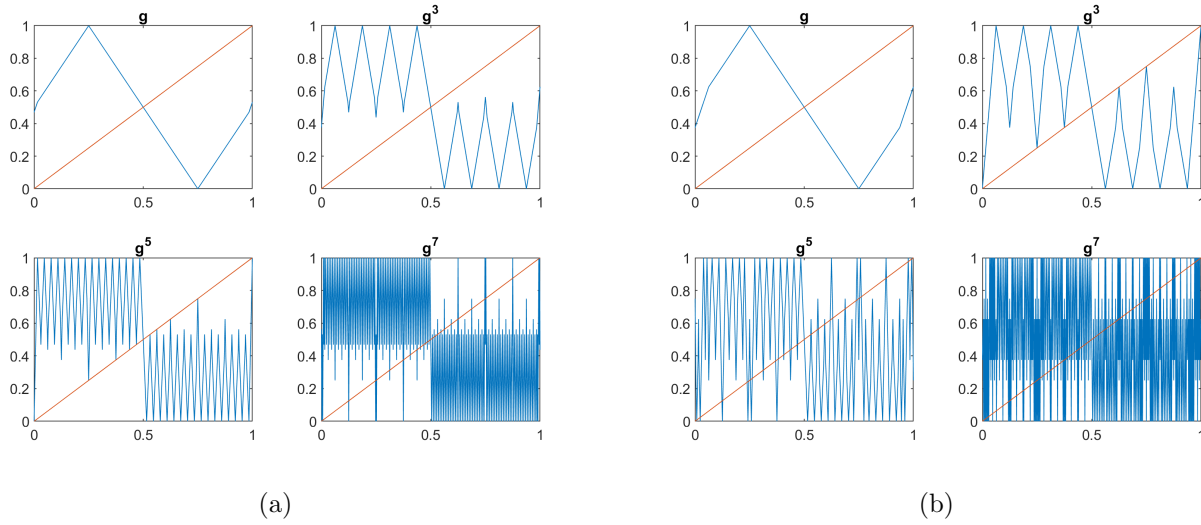


Figure 9: Two examples of $g(x)$ defined in (3). Point x_0 is a periodic point of period i if the graph of $g^i(x)$ intersects the red line representing $y = x$ at $x = x_0$, if x_0 is not a periodic point of any period $j < i$. In (a), $\delta = 2^{-5}$. Periodic points of period 3 do not exist while periodic points of periods 5 and 7 exist. In (b), $\delta = 2^{-3}$. Periodic points of period 3, 5, 7 all exist.

points of period m for all integers $m \triangleright n$. Because $3 \triangleleft n$ for any $n \neq 3$, Li-Yorke theorem is a specific case of Sharkovsky’s theorem.

From these two results, it follows that in the case where $\{\mathcal{I}_1, \mathcal{I}_2, \dots\} = \{[0, 1]\}$, there exists an odd number n_0 such that periodic points of period n exist for any odd number $n \geq n_0$, no periodic points of period n exist for any odd number $1 < n < n_0$, and periodic points of period n exist for $n = 1$ and any even number n .

Example 2. Let

$$g(x) = \begin{cases} 4x + \frac{1}{2} - \delta, & 0 \leq x < \frac{1}{2}\delta \\ 2x + \frac{1}{2}, & \frac{1}{2}\delta \leq x < \frac{1}{4} \\ -2x + \frac{3}{2}, & \frac{1}{4} \leq x < \frac{3}{4} \\ 2x - \frac{3}{2}, & \frac{3}{4} \leq x < 1 - \frac{1}{2}\delta \\ 4x - \frac{1}{2} + \delta, & 1 - \frac{1}{2}\delta \leq x \leq 1 \end{cases} \tag{3}$$

for $0 < \delta < \frac{1}{2}$. It can be shown that δ exists for any target n_0 . The smaller target value of n_0 , the larger value of δ is needed. Two examples are shown in Figure 9.

6 Entropy

Definition 12 (Entropy). The entropy of a map h on \mathcal{I} where $\mathcal{I} \subset [0, 1]$ is

$$c_{\lambda, \mathcal{I}}(h) = \int_{\mathcal{I}^0}^{\mathcal{I}^1} \log |h'(x)| d\lambda(x).$$

When $\mathcal{I} = [0, 1]$, simplify notation $c_{\lambda, [0, 1]}(h)$ to $c_{\lambda}(h)$.

Let $g \in \mathbb{G}$. Suppose that for interval \mathcal{Y} , $g^{-1}(\mathcal{Y})$ consists of m affine legs on intervals \mathcal{I}_i for $i = 1, 2, \dots, m$. Let 2^{k_i} be the absolute value of the slope of the affine segment of g on interval \mathcal{I}_i . By

definition,

$$\sum_{i=1}^m c_{\lambda, \mathcal{I}_i}(g) = \sum_{i=1}^m k_i |\mathcal{I}_i| = \sum_{i=1}^m k_i 2^{-k_i} |\mathcal{Y}|.$$

The minimum value of entropy $\sum_{i=1}^m c_{\lambda, \mathcal{I}_i}(g)$ given $|\mathcal{Y}|$ is obtained when $\{k_i\}$ solves the following integer optimization

$$\min_{k_1, \dots, k_m} \sum_{i=1}^m k_i 2^{-k_i} \tag{4}$$

$$\text{s.t. } \sum_{i=1}^m 2^{-k_i} = 1. \tag{5}$$

Lemma 15. *The solution to (4) and (5) is*

$$k_i = \begin{cases} i, & i = 1, 2, \dots, m-1 \\ m-1, & i = m. \end{cases} \tag{6}$$

Proof. We first treat k_i as continuous variables. Define the Lagrangian map

$$L(k_1, \dots, k_m, \mu) = \sum_{i=1}^m k_i 2^{-k_i} + \mu \left(\sum_{i=1}^m 2^{-k_i} - 1 \right).$$

Setting $\frac{\partial L}{\partial k_i} = 0$ leads to k_i be equal for all i , which is the interior solution and actually maximizes, rather than minimizing, the entropy. Next we check the boundary conditions. Without loss of generality, suppose that $k_1 \leq k_2 \leq \dots \leq k_m$. It is easy to see from (5) that $k_1 \geq 1$ and $k_m \leq m-1$. If $k_1 = 1$, then the original problem is reduced to $\min \sum_{i=2}^m k_i 2^{-k_i}$ subject to $\sum_{i=2}^m 2^{-k_i} = \frac{1}{2}$ and we can continue to check the boundary conditions. If $k_m = m-1$, then $k_{m-1} = m-1, k_{m-2} = m-2, k_{m-3} = m-3, \dots, k_1 = 1$. Either boundary condition leads to the same solution (6), which minimizes the entropy. \square

The set of $\{k_i\}$ (6) not only results in the minimum entropy given m but also will be shown to be the unique set that possesses the property of Lemma 17, which is a stepping stone to proving Theorem 18. Before stating Lemma 17, we take a detour to solve a dynamic matching problem.

Problem 1 (Dynamic Matching). Consider m buckets and m pumps. Pump i has a fixed pumping rate $\alpha_i > 0$. At any time, exactly one pump is pumping water into one bucket. The matching policy at time t can be represented by a permutation $\Pi_t : \{1, \dots, m\} \rightarrow \{1, \dots, m\}$. Initially all buckets are empty at time $t = 0$. The task is to find a dynamic matching policy Π_t at any time $t \in [0, 1]$ so that at $t = 1$, bucket i has $\beta_i > 0$ amount of water. By the law of conservation, $\sum_{i=1}^m \beta_i = \sum_{i=1}^m \alpha_i$.

Without loss of generality, suppose that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m$ and $\beta_1 \geq \beta_2 \geq \dots \geq \beta_m$. Not any $\{\alpha_i\}$ and $\{\beta_i\}$ have a solution. For example, if $\beta_m < \alpha_m$ or $\beta_1 > \alpha_1$, then no solution exists because the target of bucket m or 1, respectively, cannot be met. The following lemma states a necessary and sufficient condition of the existence of a solution.

Lemma 16. *A solution of Problem 1 exists if and only if $\sum_{j=1}^i (\alpha_j - \beta_j) \geq 0$ for all i .*

Proof. If $\alpha_1 - \beta_1 < 0$, then no solution exists to meet the target of bucket 1. The following assumes $\alpha_1 - \beta_1 \geq 0$. Let pump 1 only serve bucket 1. If $\alpha_1 - \beta_1 > 0$, the arrangement of pump 1 and bucket 1 is tentative, as the fraction of time that pump 1 serves bucket 1 will be adjusted in subsequent steps. The quantity $\alpha_1 - \beta_1$ represents the excess capacity due to the capacity of pump 1 in excess of the target of bucket 1.

Now consider pump 2 and bucket 2. If $\sum_{j=1}^2(\alpha_j - \beta_j) < 0$, the excess capacity $\alpha_1 - \beta_1$ is insufficient to cover the shortage of $\alpha_2 - \beta_2$. Moreover, from $\alpha_2 - \beta_2 < 0$ it follows that $\beta_2 > \alpha_2 \geq \alpha_i$ for $i > 2$. No solution exists to meet the target of bucket 2. The following assumes $\sum_{j=1}^2(\alpha_j - \beta_j) \geq 0$. There are two cases. In the first case, $\beta_2 \geq \alpha_2$. Utilize the excess capacity $\alpha_1 - \beta_1$ to cover the shortage of $\alpha_2 - \beta_2$. Specifically, let bump 2 serve bucket 2 for time interval $1 - z$ while bump 1 serves bucket 1, and then bump 1 serve bucket 2 for time interval z while bump 2 serves bucket 1, where z solves $z\alpha_1 + (1 - z)\alpha_2 = \beta_2$. Because $\alpha_2 \leq \beta_2 \leq \beta_1 \leq \alpha_1$, a unique solution z exists with $0 \leq z \leq 1$. All the arrangements regarding pump 2 and bucket 2 are final because pump 2 has no excess capacity to offer and bucket 2 has met its target. The arrangement of pump 1 and bucket 1 is still tentative if $\sum_{j=1}^2(\alpha_j - \beta_j) > 0$ because they have excess capacity to offer. In the second case, $\beta_2 < \alpha_2$. Let bump 2 only serve bucket 2. The arrangements of pump 2 and bucket 2 is also tentative as they have excess capacity to offer just like pump 1 and bucket 1. In either case, $\sum_{j=1}^2(\alpha_j - \beta_j)$ represent the accumulated excess capacity due to the total capacity of pumps 1 and 2 in excess of the total target of buckets 1 and 2.

Continue the preceding process for $i = 3, \dots, m$. If $\sum_{j=1}^i(\alpha_j - \beta_j) < 0$ for some i , then no solution exists to meet the target of bucket i , because the accumulated excess capacity from $1, 2, \dots, i - 1$, i.e., $\sum_{j=1}^{i-1}(\alpha_j - \beta_j)$, is insufficient to cover the shortage of $(\alpha_i - \beta_i)$. Otherwise, if $\beta_i \geq \alpha_i$, then utilize the accumulated excess capacity to cover the shortage of $(\alpha_i - \beta_i)$ by letting a subset of pumps $1, \dots, i - 1$, whose arrangements have so far been tentative, to serve bucket i for some fraction of time interval to meet its target while letting pump i to serve the corresponding buckets, thereby reducing the accumulated excess capacity by $(\alpha_i - \beta_i)$. If $\beta_i < \alpha_i$, then tentatively let bump i serve bucket i and note that they have excess capacity to offer, thereby increasing the accumulated excess capacity by $(\alpha_i - \beta_i)$. The process ends at $i = m$ when the accumulated excess capacity is used up to exactly cover the shortage of $(\alpha_m - \beta_m)$, because $\sum_{j=1}^m(\alpha_j - \beta_j) = 0$ from the law of conservation. \square

Now consider \mathcal{Y} where $g^{-1}(\mathcal{Y})$ consists of m affine legs. The absolute values of the slopes of the i -th affine leg is 2^i . Let $\{k_i\}$ satisfies (6) and be distinct from $\{l_i\}$. Is it possible to replace the i -th affine leg with *piecewise* affine segments on the same interval such that the absolute value of any slope is in the set of $\{2^{k_i}\}$ while preserving λ ? Lemma 17 states that not only such a replacement exists, but also the new map g_1 after the replacement is an element of \mathbb{G} and is within $\epsilon > 0$ neighborhood of the original map g . The importance of such a replacement is that the entropy of the new g_1 reaches the minimum value given m . As will be clear in Theorem 18, another map in \mathbb{G} can be constructed from g_1 to have any target value of entropy that is greater than the minimum value.

Lemma 17. *Let $g \in \mathbb{G}$ and \mathcal{Y} be an interval with dyadic endpoints. Suppose that $g^{-1}(\mathcal{Y})$ consists of m affine legs. The absolute value of the slope of the i -th leg is equal to 2^i . Partition $[0, 1]$ into $2m + 1$ intervals \mathcal{I}_j for $j = 1, 2, \dots, 2m + 1$, such that $g^{-1}(\mathcal{Y}) = \bigcup_{i=1}^m \mathcal{I}_{2i}$. Then $g_1 \in \mathbb{G}$ exists and $[0, 1]$ is partitioned into $2m + 1$ intervals \mathcal{J}_j for $j = 1, 2, \dots, 2m + 1$ such that*

- $|\mathcal{J}_{2i+1}| = |\mathcal{I}_{2i+1}|, g_1(\mathcal{J}_{2i+1}) \simeq g(\mathcal{I}_{2i+1});$
- $g_1^{-1}(\mathcal{Y})$ consists of m legs on $\{\mathcal{J}_{2i}\}$, i.e., $g_1^{-1}(\mathcal{Y}) = \bigcup_{i=1}^m \mathcal{J}_{2i};$
- $\rho(g, g_1) < \epsilon;$
- $\forall y \in \mathcal{Y}, g_1^{-1}(y) = \{x_1, \dots, x_m\}$. If none of x_i is a breakpoint, then the set of the absolute values of the slopes is $\{2^{k_i}\}$ where $\{k_i\}$ is given in (6).

Proof. Partition \mathcal{Y} evenly into 2^M intervals \mathcal{Y}_s for $s = 1, 2, \dots, 2^M$ where $2^{-M} < \frac{\epsilon}{2}$. Intervals \mathcal{I}_{2i} for $i = 1, 2, \dots, m$ are correspondingly partitioned into 2^M intervals $\{\mathcal{I}_{2i,s}\}$ where $g^{-1}(\mathcal{Y}_s) = \bigcup_{i=1}^m \mathcal{I}_{2i,s}$. First construct as follows g_2 on $g^{-1}(\mathcal{Y}_1) = \bigcup_{i=1}^m \mathcal{I}_{2i,1}$ such that $g_2(\mathcal{I}_{2i,1}^0) = g(\mathcal{I}_{2i,1}^0)$ and $g_2(\mathcal{I}_{2i,1}^1) = g(\mathcal{I}_{2i,1}^1)$, and $\forall y \in \mathcal{Y}_1$, the set of the absolute values of the derivatives at $g_2^{-1}(y)$ is $\{2^{k_i}\}$ where $\{k_i\}$ is given in (6).

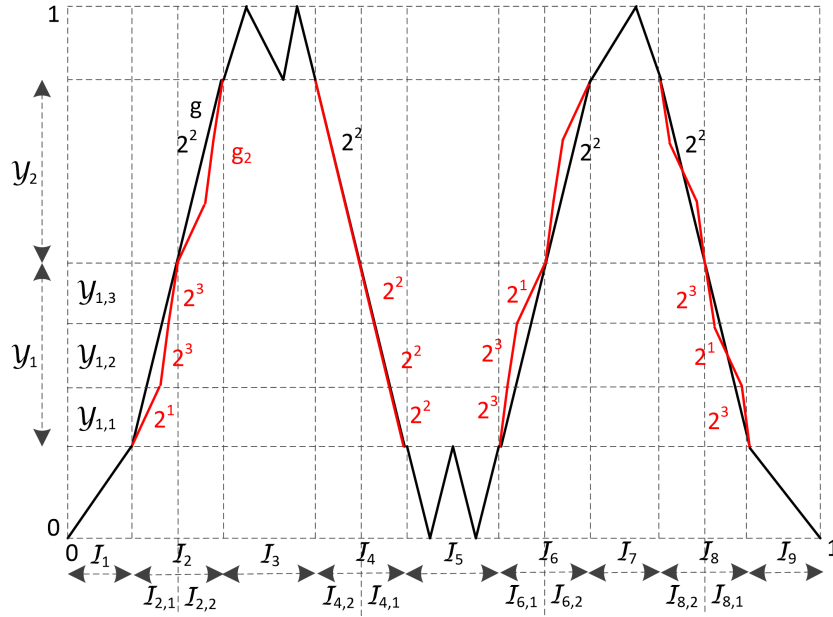


Figure 10: Construction of g_2 . The original map g is shown in black and the new map g_2 is in red. A label next to an affine segment represents the absolute value of the slope of a replacing segment in red or a replaced one in black. $M = 1, m = 4$. $\{l_i\} = \{2, 2, 2, 2\}$. $\{k_i\} = \{1, 2, 3, 3\}$. \mathcal{Y}_1 is partitioned into $\{\mathcal{Y}_{1,1}, \mathcal{Y}_{1,2}, \mathcal{Y}_{1,3}\}$. To replace $(2^2, 2^2, 2^2, 2^2)$ of g , g_2 uses $(2^1, 2^2, 2^3, 2^3)$ for $\mathcal{Y}_{1,1}$, $(2^3, 2^2, 2^3, 2^1)$ for $\mathcal{Y}_{1,2}$ and $(2^3, 2^2, 2^1, 2^3)$ for $\mathcal{Y}_{1,3}$. The same construction of g_2 is employed for \mathcal{Y}_2 with horizontal shift for continuity.

The same g_2 construction is then employed on $\mathcal{I}_{2^i,s}$ for $s = 2, 3, \dots, 2^M$ while keeping the continuity in g_2 . See Figure 10.

Partition \mathcal{Y}_1 into a number of intervals $\{\mathcal{Y}_{1,t}\}$ and perturb g as follows to obtain g_2 . On $g^{-1}(\mathcal{Y}_{1,t})$, replace the m legs of g with affine segments whose slopes keep the same signs as the original m segments of g and take the absolute values equal to a permutation of $\{2^{k_i}\}$. That is, for $i = 1, \dots, m$, replace 2^{l_i} with $2^{k_{j_t(i)}}$ where $j_t(i)$ represents a permutation of $\Pi_t : \{1, \dots, m\} \rightarrow \{1, \dots, m\}$. The length of the i -th leg on $g^{-1}(\mathcal{Y}_{1,t})$ on the x -axis is thus $2^{-k_{j_t(i)}}|\mathcal{Y}_{1,t}|$. Permutation Π_t used for different interval $\mathcal{Y}_{1,t}$ can be different. The task is to keep the total length of the i -th leg of $g^{-1}(\mathcal{Y}_1)$ on the x -axis unchanged, i.e., for all i ,

$$\sum_t 2^{-k_{j_t(i)}}|\mathcal{Y}_{1,t}| = 2^{-l_i}|\mathcal{Y}_1| = |\mathcal{I}_{2^i,1}|,$$

by employing appropriate $\{\mathcal{Y}_{1,t}\}$ partition and permutations $\{\Pi_t\}$. This problem is equivalent to Problem 1 by viewing a new, replacing affine segment as a pump at rate $\alpha_i = 2^{-k_i}$ and an original, replaced affine segment as a bucket with target $\beta_i = 2^{-l_i}$ and viewing $\{\mathcal{Y}_{1,t}\}$ partition as the partition of service time intervals and $\{\Pi_t\}$ as the matching policy in the service intervals.

In (6), $k_1 \leq k_2 \leq \dots \leq k_m$. Without loss of generality, suppose that $l_1 \leq l_2 \leq \dots \leq l_m$. Set $\{l_i\}$ can be partitioned into $\{l_{i_0+1}, \dots, l_{i_1}\}, \{l_{i_1+1}, \dots, l_{i_2}\}, \dots, \{l_{i_{n-1}+1}, \dots, l_{i_n}\}, \{l_{i_n+1}\}$ with $i_0 = 0, i_n = m - 1$ such that

$$\begin{aligned} 2^{-k_1} &= 2^{-l_{i_0+1}} + \dots + 2^{-l_{i_1}}, \\ 2^{-k_2} &= 2^{-l_{i_1+1}} + \dots + 2^{-l_{i_2}}, \\ &\dots \\ 2^{-k_n} &= 2^{-l_{i_{n-1}+1}} + \dots + 2^{-l_{i_n}}, \\ 2^{-k_{n+1}} + \dots + 2^{-k_{i_n+1}} &= 2^{-l_{i_n+1}}. \end{aligned} \tag{7}$$

As an example, suppose that $\{l_i\} = \{3, 3, 3, 3, 3, 3, 4, 4, 4, 4\}$ and $\{k_i\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 9\}$. It follows that $2^{-1} = 2^{-3} + 2^{-3} + 2^{-3} + 2^{-3}$, $2^{-2} = 2^{-3} + 2^{-3}$, $2^{-3} = 2^{-4} + 2^{-4}$, $2^{-4} = 2^{-4}$ and $2^{-5} + 2^{-6} + 2^{-7} + 2^{-8} + 2^{-9} + 2^{-9} = 2^{-4}$. It can be shown from (7) that $\sum_{j=1}^i (2^{-k_j} - 2^{-l_j}) \geq 0$ for $i = 1, \dots, m$. From Lemma 16, a solution of Problem 1 exists and can be readily derived from the constructive steps in the proof of Lemma 16. An example is illustrated in Figure 10.

The g_2 constructed above may not be an element in \mathbb{G} , because the endpoints of $\{\mathcal{Y}_{1,t}\}$ from the solution of Lemma 16 are not necessarily all dyadic. If so, replace $\{\mathcal{Y}_{1,t}\}$ with $\{\mathcal{Y}'_{1,t}\}$, which are all dyadic, while keeping permutations Π_t unchanged and $\sum_t |\mathcal{Y}'_{1,t}| = |\mathcal{Y}_1|$ to become g_1 . Choose $\{\mathcal{Y}'_{1,t}\}$ sufficiently close to $\{\mathcal{Y}_{1,t}\}$ with

$$\max_t \left| |\mathcal{Y}_{1,t}| - |\mathcal{Y}'_{1,t}| \right| < \frac{\epsilon}{2} \cdot \frac{1}{2^{\max_i(k_i, l_i)}} \cdot \frac{1}{m}. \tag{8}$$

Unlike g_2 , $|g_1^{-1}(\mathcal{Y}_1)|$ is not necessarily equal to $|g^{-1}(\mathcal{Y}_1)|$ on every leg. Because $\sum_{i=1}^m 2^{-k_i} = \sum_{i=1}^m 2^{-l_i}$, the total of $|g_1^{-1}(\mathcal{Y}_1)|$ on all the m legs is equal to that of $|g^{-1}(\mathcal{Y}_1)|$. That is,

$$\sum_t 2^{-k_{j_t(i)}} |\mathcal{Y}'_{1,t}| = 2^{-l_i} |\mathcal{Y}_1|$$

may not hold for all i ; however,

$$\sum_{i=1}^m \sum_t 2^{-k_{j_t(i)}} |\mathcal{Y}'_{1,t}| = \sum_{i=1}^m 2^{-l_i} |\mathcal{Y}_1|. \tag{9}$$

To complete the proof, partition $[0, 1]$ into $2m + 1$ intervals $\{\mathcal{J}_j\}$ such that $g_1^{-1}(\mathcal{Y}) = \bigcup_{i=1}^m \mathcal{J}_{2i}$ and $|\mathcal{J}_{2i+1}| = |\mathcal{I}_{2i+1}|$. Specifically, let $|\mathcal{J}_1| = |\mathcal{I}_1|$. For $i = 1, \dots, m$, let

$$|\mathcal{J}_{2i}| = \sum_{s=1}^{2^M} \sum_t 2^{-k_{j_t(i)}} |\mathcal{Y}'_{s,t}| \tag{10}$$

and let \mathcal{J}_{2i+1} be a horizontally shifted version of \mathcal{I}_{2i+1} to accommodate small discrepancy between $|\mathcal{I}_{2i}|$ and $|\mathcal{J}_{2i}|$. From (9) and (10),

$$\sum_{i=1}^m |\mathcal{J}_{2i}| = \sum_{i=1}^m \sum_{s=1}^{2^M} \sum_t 2^{-k_{j_t(i)}} |\mathcal{Y}'_{s,t}| = \sum_{s=1}^{2^M} \sum_{i=1}^m \sum_t 2^{-k_{j_t(i)}} |\mathcal{Y}'_{s,t}| = \sum_{s=1}^{2^M} \sum_{i=1}^m 2^{-l_i} |\mathcal{Y}_s| = \sum_{i=1}^m |\mathcal{I}_{2i}|.$$

Thus,

$$\sum_{j=1}^{2m+1} |\mathcal{J}_j| = \sum_{i=1}^m |\mathcal{J}_{2i}| + \sum_{i=0}^m |\mathcal{J}_{2i+1}| = \sum_{i=1}^m |\mathcal{I}_{2i}| + \sum_{i=0}^m |\mathcal{I}_{2i+1}| = 1.$$

Therefore, $\{\mathcal{J}_j\}$ is a valid partition of $[0, 1]$. See Figure 11 for an illustration.

Map g_1 on $\bigcup_{i=1}^m \mathcal{J}_{2i}$ has been constructed in the above. Next, let $g_1(\mathcal{J}_{2i+1}) \simeq g(\mathcal{I}_{2i+1})$ for $i = 0, \dots, m$. From (8), the maximum discrepancy between g_1 and g_2 caused by the replacement of $\{\mathcal{Y}_{1,t}\}$ on $\bigcup_{i=1}^m \mathcal{I}_{2i}$ by $\{\mathcal{Y}'_{1,t}\}$ on $\bigcup_{i=1}^m \mathcal{J}_{2i}$ and due to the horizontal shifts from $\{\mathcal{I}_{2i+1}\}$ to $\{\mathcal{J}_{2i+1}\}$ is $\frac{\epsilon}{2}$. Hence, by construction, $g_1 \in \mathbb{G}$ and $\rho(g, g_1) < \rho(g, g_2) + \rho(g_2, g_1) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. \square

Lemma 17 states that m legs of affine segments with the absolute values of the slopes equal to any set $\{2^{l_i}\}$ can be approximated by m legs of piecewise affine segments with the absolute values of the slopes $\{2^{k_i}\}$ with $\{k_i\}$ given by (6). Set of integers $\{k_i\}$ given by (6) is unique in the sense that by the analogy of Problem 1, any target amounts $\{2^{-l_i}\}$ are achievable by pumping rates $\{2^{-k_i}\}$ and the converse does not hold. Targets $\{2^{-k_i}\}$ are not achievable by any different pumping rates $\{2^{-l_i}\}$, because the difference

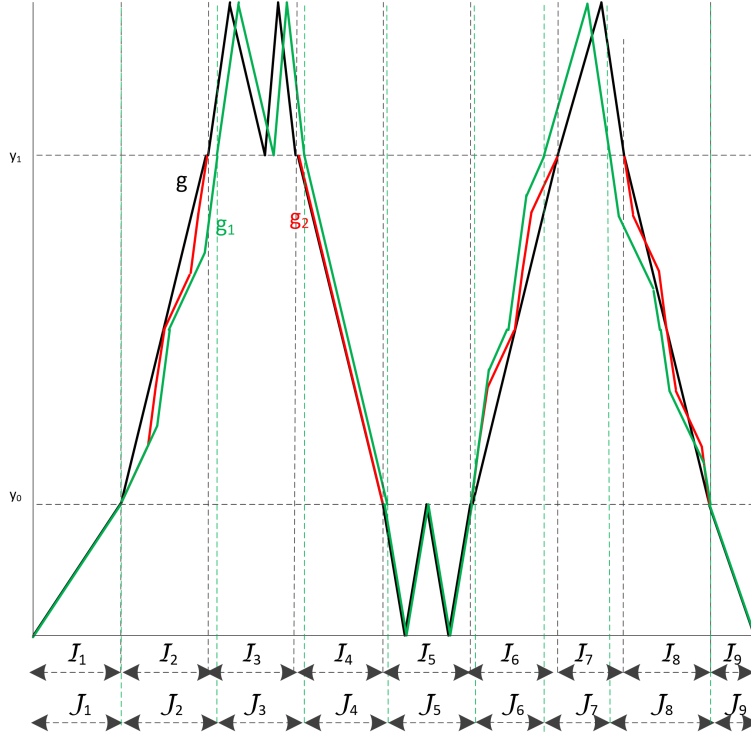


Figure 11: Construction of g_1 . Map g_1 is obtained by slightly perturbing g_2 obtained in Figure 10 such that all the breakpoints of g_1 are dyadic. Sets $\{\mathcal{I}_i\}$ and $\{\mathcal{J}_i\}$ are the partitions of $[0, 1]$ by g_2 and g_1 respectively. $|\mathcal{J}_{2i+1}| = |\mathcal{I}_{2i+1}|$ for $i = 1, \dots, m$. It is not necessary that $|\mathcal{J}_{2i}| = |\mathcal{I}_{2i}|$ for all i because all the breakpoints of g_2 are not necessarily dyadic; however, $\sum_{i=1}^m |\mathcal{I}_{2i}| = \sum_{i=1}^m |\mathcal{J}_{2i}|$.

between the maximum and minimum achievable amounts is upper-bounded by $\max_i 2^{-l_i} - \min_i 2^{-l_i}$, which is smaller than what is required by the targets $\max_i 2^{-k_i} - \min_i 2^{-k_i} = 2^{-1} - 2^{-m+1}$.

The slope of an affine segment of $g \in \mathbb{G}$ has to be in the form of $\pm 2^k$ for non-negative integer k . If g is required to be LEO, then k must be positive, and as a result, the entropy must be at least equal to 1. If g has m legs on $[0, 1]$, from Lemma 15, the minimum value of entropy $c_\lambda(g)$ is given by

$$c_{\min}(m) = \sum_{i=1}^{m-1} i2^{-i} + (m-1)2^{-(m-1)}. \tag{11}$$

It is easy to confirm that $c_{\min}(m)$ is an increasing sequence of m and $\lim_{m \rightarrow \infty} c_{\min}(m) = 2$. Therefore, for any m ,

$$c_{\min}(m) < 2. \tag{12}$$

Theorem 18. For any $c \in [2, \infty)$ and $\epsilon > 0$, the subset of Markov LEO maps in \mathbb{G} whose entropy is within ϵ of c is dense in $C(\lambda)$.

Proof. Let $h \in C(\lambda)$. We will show that $g \in \mathbb{G}$ exists such that g is Markov and LEO, $\rho(h, g) < \epsilon$ and $|c - c_\lambda(g)| < \epsilon$.

By Theorem 10, $g_0 \in \mathbb{G}$ exists such that g_0 is Markov and LEO, and $\rho(h, g_0) < \frac{\epsilon}{3}$. Partition $[0, 1]$ into n intervals $\{\mathcal{Y}_i\}$ such that $g_0^{-1}(\mathcal{Y}_i)$ has m_i legs for $i = 1, \dots, n$. Applying Lemma 17 to all the n intervals, $g_1 \in \mathbb{G}$ exists such that $\rho(g_0, g_1) < \frac{\epsilon}{3}$ and the entropy of g_1 is given by

$$c_\lambda(g_1) = \sum_{i=1}^n c_{\min}(m_i)\lambda(\mathcal{Y}_i) \leq 2 \sum_{i=1}^n \lambda(\mathcal{Y}_i) = 2. \tag{13}$$

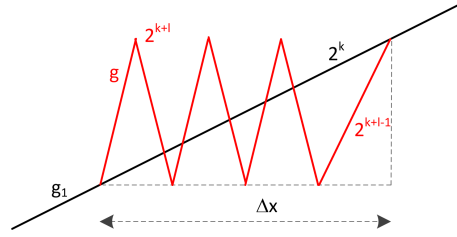


Figure 12: Increase entropy by replacing an affine segment (black) with a window perturbation (red). In this figure, $l = 3$.

The inequality is because of (12).

Consider any $c \geq 2$. If $|c - c_\lambda(g_1)| \leq \epsilon$, then let $g = g_1$ and the proof is complete. Otherwise, $|c - c_\lambda(g_1)| > \epsilon$. From (13), $c - c_\lambda(g_1) > \epsilon$. Select any affine segment of g_1 of dyadic endpoints and dyadic length Δx and replace it with a window perturbation to obtain g , as shown in Figure 12. Specifically, suppose that the selected affine segment is of slope 2^k . Let $\Delta x < 2^{-k} \cdot \frac{\epsilon}{3}$. Thus, $\rho(g_1, g) < \frac{\epsilon}{3}$. Hence, $\rho(h, g) < \rho(h, g_0) + \rho(g_0, g_1) + \rho(g_1, g) < \epsilon$.

The window perturbation consists of $2^l - 1$ legs, where each of the first $2^l - 2$ legs is of slope of absolute value 2^{k+l} and the last leg is of slope of absolute value 2^{k+l-1} , for integer l . The increase in entropy from g_1 to g on Δx is given by $\Delta x \cdot (l - 2^{-l+1})$. Thus,

$$c_\lambda(g) = c_\lambda(g_1) + \Delta x \cdot (l - 2^{-l+1}).$$

Select Δx sufficiently small and l sufficiently large such that

$$\left| \Delta x \cdot (l - 2^{-l+1}) - (c - c_\lambda(g_1)) \right| < \epsilon.$$

Hence, $|c_\lambda(g) - c| < \epsilon$, which completes the proof. □

The authors in [3, Proposition. 21] prove that for every $c \in (0, \infty)$, the subset of Markov LEO $PA(\lambda)$ is dense in $C(\lambda)$ with $c_\lambda(h) = c$. Comparison with Theorem 18 shows that because the breakpoints in \mathbb{G} are constrained to be dyadic, $c_\lambda(g)$ can only be within any ϵ neighborhood of an entropy target c but is not guaranteed to be exactly equal to c . Moreover, to be LEO, the absolute value of the slope of any affine segment can only take a discrete value of 2^k for positive integer k in \mathbb{G} , but can be any real number no smaller than 1 in $PA(\lambda)$. Given a target map h , the smallest entropy value achievable by the dense subset of \mathbb{G} is thus greater than what is achievable by that of $PA(\lambda)$.

7 Decomposition

Recall that \mathbb{F} is generated by two generator maps. Theorem 33 in this section is to show that any map in \mathbb{G} can be expressed as a composition of a finite number of basic maps in \mathbb{G} and the generators in \mathbb{F} . To this end, first Theorem 28 shows that any map in \mathbb{G} is a composition of maps in \mathbb{F} and window perturbations, and then all window perturbations are shown to be generated by a few basic maps in \mathbb{G} . On the other hand, Theorem 35 shows that unlike \mathbb{F} , \mathbb{G} is not finitely generated.

First, consider type I breakpoints.

Lemma 19. *Let $\{\mathcal{I}_1 < \dots < \mathcal{I}_{2n+1}\}$ be a partition of $[0, 1]$. Let $g \in \mathbb{G}$. Suppose that for $i = 1, \dots, n$, g is an affine segment on \mathcal{I}_{2i} whose slope is $(-1)^{p_i} 2^{k_i}$, and $\mathcal{Y} = g(\mathcal{I}_{2i})$ is the same for all i . If integers $\{l_i\}$*

exist such that

$$\sum_{i=1}^n 2^{-k_i} = \sum_{i=1}^n 2^{-l_i}, \tag{14}$$

then $f_1 \in \mathbb{F}$, $g_1 \in \mathbb{G}$ and another partition of $[0, 1]$ $\{\mathcal{J}_1 < \dots < \mathcal{J}_{2n+1}\}$ exist such that composition $g_1(f_1(x)) = g(x)$ for any $x \in [0, 1]$, $g_1(\mathcal{J}_j) \simeq g(\mathcal{I}_j)$ for any j , and

- for odd j , $|\mathcal{J}_j| = |\mathcal{I}_j|$;
- for even $j = 2i$, g_1 on \mathcal{J}_{2i} is an affine segment whose slope is equal to $(-1)^{p_i} 2^{l_i}$.

Proof. The set of intervals $\{\mathcal{J}_j\}$ is completely defined once their lengths are defined. Specifically, let

$$|\mathcal{J}_j| = \begin{cases} |\mathcal{I}_j|, & \text{for odd } j; \\ |\mathcal{Y}| \cdot 2^{-l_i}, & \text{for even } j = 2i. \end{cases} \tag{15}$$

The endpoints of any interval \mathcal{J}_j are dyadic by construction. To show the above partition is a valid one, note that for $i = 1, 2, \dots, n$,

$$\begin{aligned} |\mathcal{I}_{2i}| &= |\mathcal{Y}| \cdot 2^{-k_i}, |\mathcal{J}_{2i}| = |\mathcal{Y}| \cdot 2^{-l_i} \\ \xrightarrow{\text{By (14)}} \sum_{i=1}^n |\mathcal{I}_{2i}| &= \sum_{i=1}^n |\mathcal{J}_{2i}| \implies \sum_{j=1}^{2n+1} |\mathcal{J}_j| = \sum_{j=1}^{2n+1} |\mathcal{I}_j| = 1. \end{aligned}$$

Construct g_1 as follows. With odd j , for $x \in \mathcal{J}_j$, let $g_1(x) = g(x - d_j)$, where $d_j = \mathcal{J}_j^0 - \mathcal{I}_j^0$. For even $j = 2i$, the graph of $g_1(x)$ on \mathcal{J}_j is the affine segment that connects the right endpoint of g_1 on \mathcal{J}_{j-1} and the left endpoint of g_1 on \mathcal{J}_{j+1} . Thus by construction (15), the slope of the affine segment is $(-1)^{p_i} 2^{l_i}$. Moreover, by (14), g_1 is λ -preserving. Hence, $g_1 \in \mathbb{G}$.

Construct f_1 as follows. Set $f_1(0) = 0$. For $j = 1, 2, \dots, 2n + 1$ and $x \in \mathcal{I}_j$, the slope of f_1 is set to 1 for odd j and to $2^{k_i - l_i}$ for even $j = 2i$. By construction, the breakpoints of f_1 are dyadic and all the slopes are in the form of 2^m for integer m . To validate that $f_1 \in \mathbb{F}$, note that

$$\begin{aligned} |\mathcal{I}_{2i}| &= |\mathcal{Y}| \cdot 2^{-k_i} \implies |f_1(\mathcal{I}_{2i})| = |\mathcal{Y}| \cdot 2^{-k_i} \cdot 2^{k_i - l_i} = |\mathcal{Y}| \cdot 2^{-l_i} \\ \xrightarrow{\text{By (14)}} \sum_{i=1}^n |f_1(\mathcal{I}_{2i})| &= \sum_{i=1}^n |\mathcal{I}_{2i}| \implies \sum_{j=1}^{2n+1} |f_1(\mathcal{I}_j)| = \sum_{j=1}^{2n+1} |\mathcal{I}_j| = 1 \end{aligned}$$

Finally, to show that $g_1(f_1) = g$, note that by construction of f_1 and g_1 that for $j = 1, 2, \dots, 2n + 1$,

$$f_1(\mathcal{I}_j) \simeq \mathcal{J}_j, g_1(\mathcal{J}_j) \simeq g(\mathcal{I}_j) \implies g_1(f_1(\mathcal{I}_j)) \simeq g(\mathcal{I}_j)$$

Hence, $g_1(f_1) = g$. □

Corollary 20. Let $\{\mathcal{I}'_1 < \dots < \mathcal{I}'_{2n+1}\}$ be a partition of $[0, 1]$. Let $g \in \mathbb{G}$. Suppose that for $i = 1, \dots, n$, g is a piecewise affine segment containing a single type I breakpoint A_i on \mathcal{I}'_{2i} . Suppose that $\mathcal{Y} = g(\mathcal{I}'_{2i})$ is the same for all i and that $A_{i,y}$ is the same for $i = 1, \dots, n$. Let $(-1)^{p_i} 2^{l_i}$ be the slope of the affine segment on $g^{-1}([\mathcal{Y}^0, A_{i,y}]) \cap \mathcal{I}'_{2i}$ and $(-1)^{p_i} 2^{k_i}$ be the slope of the affine segment on $g^{-1}([A_{i,y}, \mathcal{Y}^1]) \cap \mathcal{I}'_{2i}$. If (14) holds, then $f_1 \in \mathbb{F}$, $g_1 \in \mathbb{G}$ and another partition of $[0, 1]$ $\{\mathcal{J}'_1 < \dots < \mathcal{J}'_{2n+1}\}$ exist such that composition $g_1(f_1(x)) = g(x)$ for any $x \in [0, 1]$,

- for odd j , $|\mathcal{I}'_j| = |\mathcal{J}'_j|$ and $g_1(\mathcal{J}'_j) \simeq g(\mathcal{I}'_j)$;
- for even j , g_1 on \mathcal{J}'_j is an affine segment.

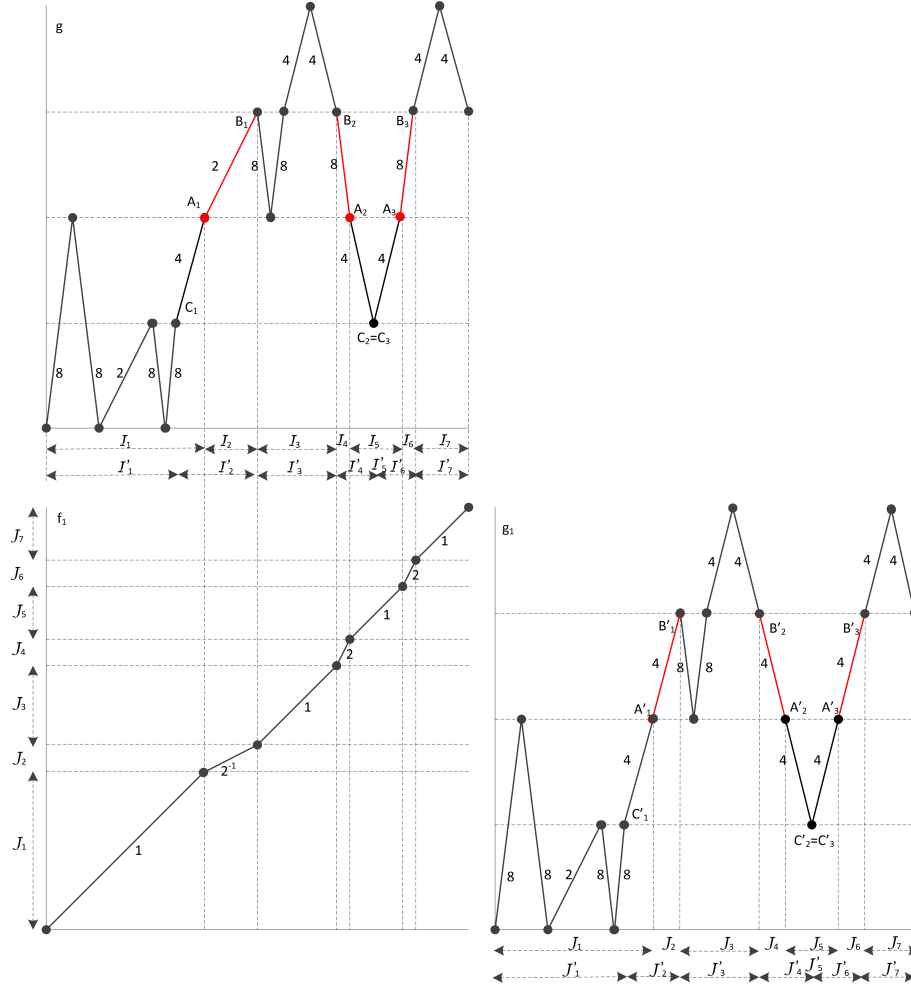


Figure 13: Use of Lemma 19 and Corollary 20. The red segments in g are replaced by those in g_1 where $g = g_1 \circ f_1$. As a result, type I breakpoints A_1, A_2, A_3 of g are eliminated in g_1 because the left and right derivatives are the same at A'_1, A'_2, A'_3 . The number next to an affine segment represents the absolute value of the slope.

Proof. By Lemma 19, one can replace the affine segment of g on $g^{-1}([A_{i,y}, \mathcal{Y}^1]) \cap \mathcal{I}'_{2i}$ with another one whose slope is equal to $(-1)^{p_i} 2^{l_i}$ for $i = 1, 2, \dots, n$, while keeping the slope of affine segment unchanged on $g^{-1}([\mathcal{Y}^0, A_{i,y}]) \cap \mathcal{I}'_{2i}$. As a result, g_1 on $[\mathcal{I}'_{2i}, \mathcal{I}'_{2i+1}]$ is an affine segment with no type I breakpoint inside. □

Figure 13 illustrates the use of Lemma 19 and Corollary 20.

By Corollary 20, g can be generated by g_1 , which eliminates certain type I breakpoints of g . In particular, the following corollary holds.

Corollary 21. *Let $\{\mathcal{I}_i\}$ for $i = 1, \dots, m$ be a set of intervals with mutually disjoint interiors. Let $g \in \mathbb{G}$. If g is monotone on \mathcal{I}_i for all i , $\mathcal{Y} = g(\mathcal{I}_i)$ is the same for all i , and $g^{-1}(\mathcal{Y}) = \bigcup_{i=1}^m \mathcal{I}_i$, then all the type I breakpoints in the interiors of $\{\mathcal{I}_i\}$ can be eliminated.*

Proof. $\forall y \in \mathcal{Y}$, $g^{-1}(\mathcal{Y})$ consists of one point on each \mathcal{I}_i . If the derivative exists at all points of $g^{-1}(y)$, then (2) holds and so does (14) with a common set of $\{l_i\}$ for any y . The set of the affine segments in the neighborhood of $g^{-1}(y)$, i.e., $g^{-1}([y - \delta, y + \delta])$ for some $\delta > 0$, whose slopes have absolute values equal to $2^{k_i(y)}$ on \mathcal{I}_i can be replaced by affine segments with absolute values of slopes equal to 2^{l_i} with

$\sum_{i=1}^m 2^{-l_i} = 1$. Set $\{2^{l_i}\}$ remain the same for all $y \in \mathcal{Y}$. Thus, by Corollary 20, the piecewise affine segment on each \mathcal{I}_i is replaced by an affine segment and any breakpoint in the interior of \mathcal{I}_i is eliminated. \square

Corollary 21 covers the case of $g^{-1}(g(\bigcup_{i=1}^m \mathcal{I}_i)) = \bigcup_{i=1}^m \mathcal{I}_i$. The opposite case is where $\forall y \in g(\bigcup_{i=1}^m \mathcal{I}_i)$, $g^{-1}(y) \not\subset \bigcup_{i=1}^m \mathcal{I}_i$, which is addressed next.

Lemma 22. *Let $\{\mathcal{I}_1 < \dots < \mathcal{I}_{2m+1}\}$ be a partition of $[0, 1]$. Suppose that $g \in \mathbb{G}$ is an affine segment on \mathcal{I}_{2i} and $\mathcal{Y} = g(\mathcal{I}_{2i})$ is the same for $i = 1, \dots, m$. If $\forall y \in \mathcal{Y}$, $g^{-1}(y) \not\subset \bigcup_{i=1}^m \mathcal{I}_i$, then $f_1 \in \mathbb{F}$, $g_1 \in \mathbb{G}$ and another partition of $[0, 1]$ $\{\mathcal{J}_1 < \dots < \mathcal{J}_{2m+1}\}$ exist such that composition $g_1(f_1(x)) = g(x)$ for any $x \in [0, 1]$, $g_1(\mathcal{J}_{2i}) \simeq g(\mathcal{I}_{2i})$ and g_1 is an affine segment on \mathcal{J}_{2i} with the absolute value of the slope being 2^{k_i} where $\sum_{i=1}^m 2^{-k_i} = 2^{-K}$ for some positive integer K .*

Proof. Partition \mathcal{Y} into intervals $\{\mathcal{Y}_j\}$, $j = 1, 2, \dots, n$, such that no breakpoint exists whose y -coordinate falls in the interior of any \mathcal{Y}_j , i.e., \nexists breakpoint B such that $B_y \in (\mathcal{Y}_j^0, \mathcal{Y}_j^1)$. Consider $g^{-1}(\mathcal{Y}_j)$. Let $g^{-1}(\mathcal{Y}_j) = \{\mathcal{I}_{j,1}, \mathcal{I}_{j,2}, \dots, \mathcal{I}_{j,m}, \mathcal{I}_{j,m+1}, \dots, \mathcal{I}_{j,m+n_j}\}$ with mutually disjoint interiors, where interval $\mathcal{I}_{j,i} \subset \mathcal{I}_{2i}$ for $i = 1, \dots, m$. By the hypothesis of the lemma, $n_j \geq 1$. Because $\mathcal{Y} = \bigcup_{j=1}^n \mathcal{Y}_j$, it follows that for $i = 1, \dots, m$,

$$\bigcup_{j=1}^n \mathcal{I}_{j,i} = \mathcal{I}_{2i}. \tag{16}$$

The graph of g is affine on every $\mathcal{I}_{j,i}$ because no breakpoint exists on \mathcal{Y}_j° . Let $2^{k_{j,i}}$ be the absolute value of the slope of the affine segment on $\mathcal{I}_{j,i}$. Because g is λ -preserving,

$$\sum_{i=1}^{m+n_j} 2^{-k_{j,i}} = 1. \tag{17}$$

Let

$$\sum_{i=1}^m 2^{-k_{j,i}} = L \cdot 2^{-K}, \tag{18}$$

where L is an odd integer. Because the graph of g is an affine segment on \mathcal{I}_{2i} , $k_{j,i}$ does not depend on j when $i = 1, \dots, m$, which is the reason that L, K do not have subscript j in (18).

Assume that $L > 1$. The following iterative procedure is employed to decrement L by adjusting $k_{j,i}$ where $1 \leq i \leq m + n_j$ while satisfying (17) so that eventually $L = 1$.

From (18), $\sum_{i=1}^m 2^{-k_{j,i}+K} = L$. Focus on $k_{j,i}$ that is greater than or equal to K and arrange them in an increasing order. Add such $2^{-k_{j,i}+K}$ terms one-by-one until the sum reaches 1. Therefore, a subset of $\{1, 2, \dots, m\}$, denoted by Φ_1 , exists such that $\sum_{i \in \Phi_1} 2^{-k_{j,i}} = 2^{-K}$. Because $L - 1$ is even, let $\sum_{i=1, i \notin \Phi_1}^m 2^{-k_{j,i}} = (L - 1) \cdot 2^{-K} = L' \cdot 2^{-K'}$, where $0 < K' < K$ and L' is odd. $L' < L$. Similarly, a subset of $\{1, 2, \dots, m\} \setminus \Phi_1$, denoted by Φ_2 , exists such that $\sum_{i \in \Phi_2} 2^{-k_{j,i}} = 2^{-K'}$. Now let

$$k'_{j,i} = \begin{cases} k_{j,i} + K' + 1 - K, & i \in \Phi_1; \\ k_{j,i} + 1, & i \in \Phi_2; \\ k_{j,i}, & i \in \{1, 2, \dots, m\} \setminus (\Phi_1 \cup \Phi_2). \end{cases} \tag{19}$$

Therefore,

$$\sum_{i=1}^m 2^{-k'_{j,i}} = \left(\sum_{i \in \Phi_1} + \sum_{i \in \Phi_2} + \sum_{i \in \{1, 2, \dots, m\} \setminus (\Phi_1 \cup \Phi_2)} \right) 2^{-k'_{j,i}} = 2^{-K} \cdot 2^{K-K'-1} + 2^{-K'} \cdot 2^{-1} + (L' - 1) \cdot 2^{-K'} = L' \cdot 2^{-K'}.$$

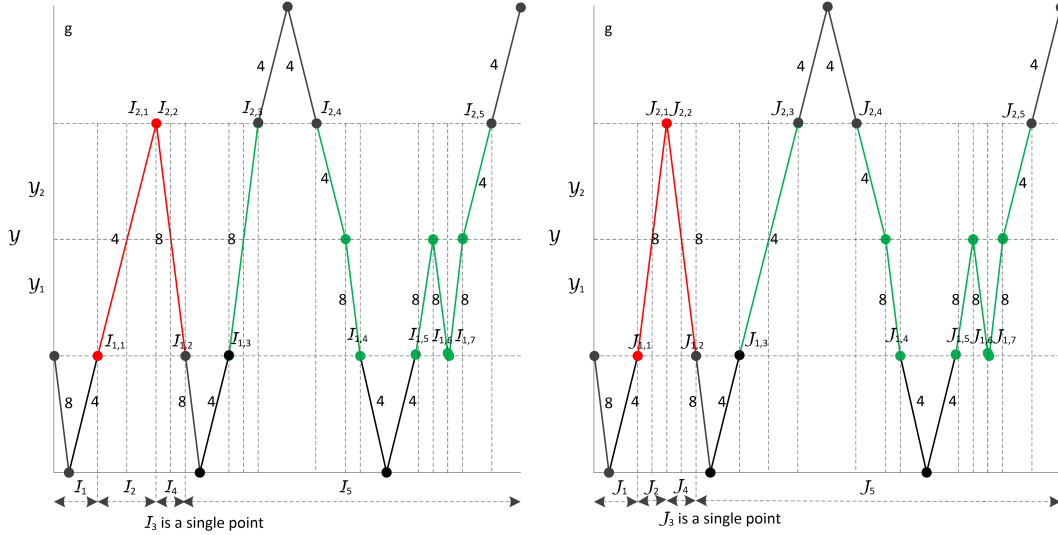


Figure 14: Illustration of the proof of Lemma 22. The graphs of g on $\{\mathcal{I}_{2i}\}$ and g_1 on $\{\mathcal{J}_{2i}\}$ are drawn in red. The graphs of g on $g^{-1}(\mathcal{Y}) \cap ([0, 1] \setminus \bigcup_{i=1}^m \mathcal{I}_{2i})$ and g_1 on $g_1^{-1}(\mathcal{Y}) \cap ([0, 1] \setminus \bigcup_{i=1}^m \mathcal{J}_{2i})$ are drawn green. $m = 2, n = 2, n_1 = 7, n_2 = 5$. g on $\bigcup_{i=1}^m \mathcal{I}_{2i}$ satisfies $2^{-2} + 2^{-3} = 3 \cdot 2^{-3}$ in (18). g_1 on $\bigcup_{i=1}^m \mathcal{J}_{2i}$ satisfies $2^{-3} + 2^{-3} = 1 \cdot 2^{-2}$ in (21).

On the other hand, $\sum_{i=m+1}^{m+n_j} 2^{-k_{j,i}} = 1 - L \cdot 2^{-K}$. Similarly, a subset of $\{m+1, m+2, \dots, m+n_j\}$, denoted by Ψ , exists such that $\sum_{i \in \Psi} 2^{-k_{j,i}} = 2^{-K}$. Now let

$$k'_{j,i} = \begin{cases} k_{j,i} - 1, & i \in \Psi; \\ k_{j,i}, & i \in \{m+1, m+2, \dots, m+n_j\} \setminus \Psi. \end{cases} \tag{20}$$

Therefore,

$$\sum_{i=m+1}^{m+n_j} 2^{-k'_{j,i}} = \left(\sum_{i \in \Psi} + \sum_{i \in \{m+1, m+2, \dots, m+n_j\} \setminus \Psi} \right) 2^{-k'_{j,i}} = 2^{-K} \cdot 2^1 + (1 - L \cdot 2^{-K} - 2^{-K}) = 1 - L' \cdot 2^{-K'}$$

From $\{k_{j,i}\}$ to $\{k'_{j,i}\}$, L drops to L' . The above process continues until $L = 1$. Because $k_{j,i}$ does not depend on j when $i = 1, 2, \dots, m$, neither does $k'_{j,i}$ in (19). This property remains in the process.

When the process ends, for any j

$$\sum_{i=1}^m 2^{-k''_{j,i}} = 2^{-K''} \tag{21}$$

Because

$$\sum_{i=1}^{m+n_j} 2^{-k''_{j,i}} = \sum_{i=1}^{m+n_j} 2^{-k_{j,i}} = 1, \tag{22}$$

apply Lemma 19 for all j one-by-one. Then $f_1 \in \mathbb{F}, g_1 \in \mathbb{G}$ and another partition of $[0, 1]$ $\{\mathcal{J}_1 < \dots < \mathcal{J}_{2m+1}\}$ exist such that composition $g_1(f_1(x)) = g(x)$ for any $x \in [0, 1]$. Just like (16), for $i = 1, \dots, m$, \mathcal{J}_{2i} can be decomposed to $\mathcal{J}_{2i} = \bigcup_{j=1}^n \mathcal{J}_{j,i}$ such that $g_1(\mathcal{J}_{j,i}) \simeq g(\mathcal{I}_{j,i})$. Thus, $g_1(\mathcal{J}_{2i}) \simeq g(\mathcal{I}_{2i})$. The graph of g_1 is an affine segment on $\mathcal{J}_{j,i}$ with the absolute value of the slope being $2^{k''_{j,i}}$, which does not depend on j when $i = 1, 2, \dots, m$. Thus, g_1 is an affine segment in every \mathcal{J}_{2i} . The absolute values of the slopes of these affine segments for $i = 1, \dots, m$ satisfies (21). This completes the proof. \square

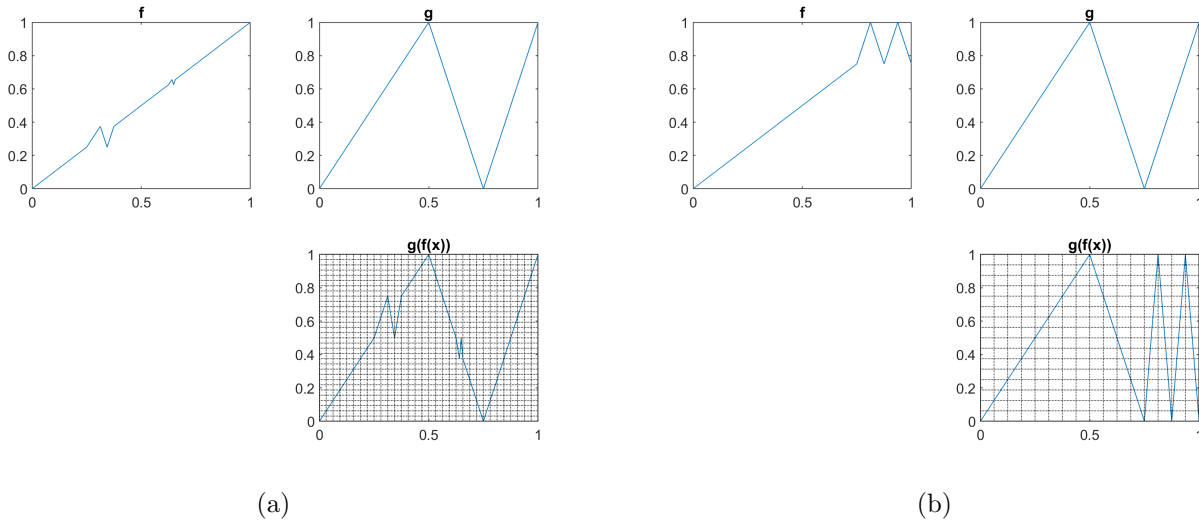


Figure 15: Use of Lemma 23: (a) the case of odd m and (b) the case of even m . A few type II breakpoints in $g(f(x))$ are eliminated in $g(x)$.

Figure 14 shows an example to illustrate the proof of Lemma 22.

Next, consider type II breakpoints. Let w be an m -fold window perturbation on interval \mathcal{I} and $w(x) = x$ or $w(x) = 1 - x$ if $x \in [0, 1] \setminus \mathcal{I}$ to continuously connect the end points of the window perturbation. Abusing the terminology, we call w a window perturbation too. Let 2^{k_i} be the absolute value of the slope of the i -th leg of the m -fold window perturbation w for $i = 1, 2, \dots, m$. $\sum_{i=1}^m 2^{-k_i} = 1$ to be λ -preserving.

Suppose that g_1 on \mathcal{I} is an affine segment, whose slope has an absolute value of 2^K . Then composition $g_1(w_1)$ is identical to g_1 on $[0, 1] \setminus \mathcal{I}$. On \mathcal{I} the affine segment of g_1 is replaced by an m -fold window perturbation, whose i -th leg has an absolute value of the slope 2^{k_i+K} . The ratio of the slopes of any two legs of $g_1(w_1)$ is the same as that of the corresponding legs of w_1 . Lemma 23 follows.

Lemma 23. *Suppose that g on interval \mathcal{I} is an m -fold window perturbation. Let 2^{k_i+K} be the absolute value of the slope of the i -th leg. If $\sum_{i=1}^m 2^{-k_i} = 1$, then $g_1 \in \mathbb{G}$ and an m -fold window perturbation map w_1 exist such that composition $g_1(w_1(x)) = g(x)$ for any $x \in [0, 1]$, $g_1([0, \mathcal{I}^0]) \simeq g([0, \mathcal{I}^0])$, $g_1([\mathcal{I}^1, 1]) \simeq g([\mathcal{I}^1, 1])$, and g_1 is an affine segment on \mathcal{I} whose slope has the absolute value of 2^K .*

Figure 15 shows the use of Lemma 23. Interval \mathcal{I} can be anywhere on $[0, 1]$ for odd m . For even m , \mathcal{I} must cover at least one endpoint; that is, $\mathcal{I}^0 = 0$ or $\mathcal{I}^1 = 1$.

A special case of Lemma 23 is when $K = 0$. Because $\sum_{i=1}^m 2^{-k_i} = 1$, $\forall y \in g(\mathcal{I})$, $g^{-1}(y) \subset \mathcal{I}$. Thus, $g^{-1}(g(\mathcal{I})) = \mathcal{I}$. This case is similar to what Corollary 21 covers. On the other hand, just like Lemma 22, Lemma 24 covers the case where $\forall y \in g(\bigcup_{i=1}^m \mathcal{I}_i)$, $g^{-1}(y) \not\subset \bigcup_{i=1}^m \mathcal{I}_i$. Specifically, in Lemma 22 when \mathcal{I}_{2i} , for $i = 1, \dots, m$, are adjacent, i.e., each of $\mathcal{I}_3, \mathcal{I}_5, \dots, \mathcal{I}_{2m-1}$ is reduced to a single point, the m affine segments form a m -fold window perturbation and can be further simplified as stated in Lemma 24.

Lemma 24. *Suppose that in Lemma 22, $\bigcup_{i=1}^m \mathcal{I}_{2i}$ is an interval, denoted by \mathcal{I} . That is, g is an m -fold window perturbation on \mathcal{I} . Then $f_1 \in \mathbb{F}$, $g_1 \in \mathbb{G}$, an m -fold window perturbation map w_1 and another partition of $[0, 1]$ $\{\mathcal{J}_1 < \dots < \mathcal{J}_{2m+1}\}$ exist such that composition $g_1(w_1(f_1(x))) = g(x)$ for any $x \in [0, 1]$, and $\bigcup_{i=1}^m \mathcal{J}_{2i}$ is an interval on which g_1 is an affine segment.*

Proof. Maps f_1, g_2 and interval partition $\{\mathcal{J}_1 < \dots < \mathcal{J}_{2m+1}\}$ are obtained by Lemma 22 such that $g_2(f_1) = g$. Each of $\mathcal{J}_3, \mathcal{J}_5, \dots, \mathcal{J}_{2m-1}$ is reduced to a single point just like $\mathcal{I}_3, \mathcal{I}_5, \dots, \mathcal{I}_{2m-1}$. Thus, $\bigcup_{i=1}^m \mathcal{J}_{2i}$ is an interval, denoted by \mathcal{J} . In the proof of Lemma 22, g_2 on \mathcal{J} is an m -fold window

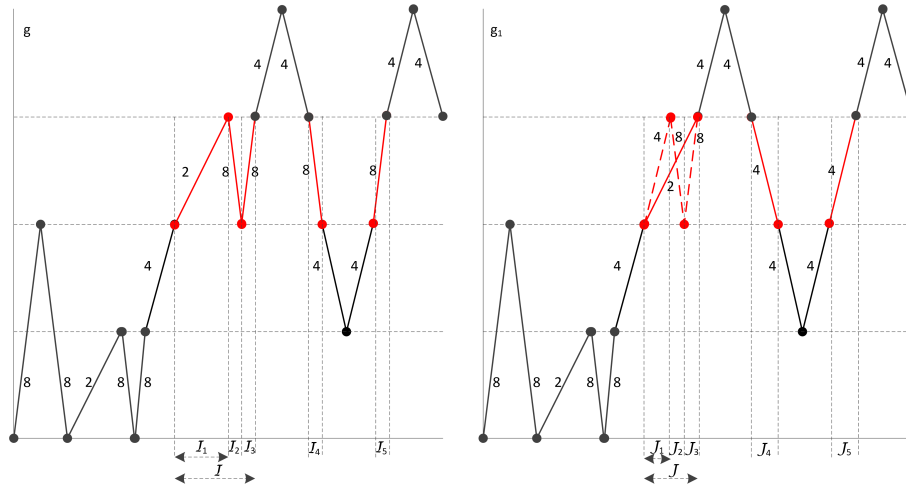


Figure 16: Use of Lemma 24. The three solid red affine segments in the right figure represent g_1 . The leftmost one is replaced by three dashed red segments due to a window perturbation w_1 with $m = 3$. The resultant five red segments, dashed and solid, are horizontally adjusted by f_1 to generate g in the left figure. Two type II breakpoints of g are eliminated in g_1 .

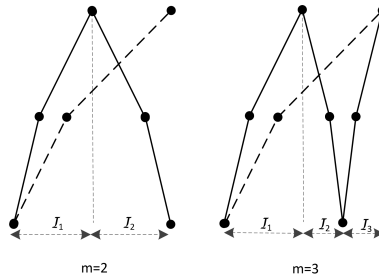


Figure 17: Illustration of generalized m -fold window perturbation

perturbation, just like g on \mathcal{I} . The difference between the two m -fold window perturbations is that the slopes of their legs are described in (18) and (21) respectively. Now let w_1 be an m -fold window perturbation on \mathcal{J} where the slope of the i -th leg is $(-1)^{i+1}2^{k''_{j,i}-K''}$ for $i = 1, 2, \dots, m$. Modify g_2 to become g_1 where the m -fold window perturbation on \mathcal{J} of g_2 is replaced by an affine segment of g_1 whose slope has an absolute value of $2^{K''}$. The sign of the slope is such that g_1 is continuous at the endpoints of \mathcal{J} . By Lemma 23, $g_2 = g_1(w_1)$. Therefore, $g_1(w_1(f_1)) = g$. \square

Figure 16 illustrates the use of Lemma 24. By Lemma 24, g can be generated by g_1 by eliminating $m - 1$ type II breakpoints.

In Lemma 23 and Lemma 24, g has to be an m -fold window perturbation on \mathcal{I} . The m -fold window perturbation consists of m legs each of which is an affine segment. One can generalize the notion of m -fold window perturbation such that it consists of m legs each of which itself consists of piecewise affine segments. The more precise definition is given below.

Definition 13 (Generalized Window Perturbation). A generalized m -fold window perturbation g on \mathcal{I} is defined as follows. Suppose that \mathcal{I} is partitioned into $\{\mathcal{I}_1 < \mathcal{I}_2 < \dots < \mathcal{I}_m\}$. $g(\mathcal{I}_i) \simeq g(\mathcal{I}_j)$ if both i and j are either odd or even and $g(\mathcal{I}_i) \simeq g(\hat{\mathcal{I}}_j)$ if one of i, j is even and the other is odd where $\hat{\mathcal{I}}_j$ represents \mathcal{I}_j flipped horizontally. Each \mathcal{I}_i is referred to as a component interval of \mathcal{I} .

Figure 17 provides two examples of the generalized m -fold window perturbations, one for an even m and the other for an odd m .

Corollary 25. *Lemma 23 and Lemma 24 hold if g is a generalized m -fold window perturbation instead of an m -fold window perturbation on interval \mathcal{I} , except that in the conclusion g_1 is a piecewise affine segment, instead of an affine segment, on \mathcal{J} where $g_1(\mathcal{J}) \simeq g(\mathcal{I}_0)$ or $g_1(\mathcal{J}) \simeq g(\hat{\mathcal{I}}_0)$ with \mathcal{I}_0 being one component interval of \mathcal{I} . Whether $g_1(\mathcal{J}) \simeq g(\mathcal{I}_0)$ or $g_1(\mathcal{J}) \simeq g(\hat{\mathcal{I}}_0)$ depends on the continuity of g_1 .*

Recall that Corollary 21 and Lemma 22 consider two cases respectively: either $\forall y \in g(\mathcal{I}), g^{-1}(y) \subset \mathcal{I}$ or $\forall y \in g(\mathcal{I}), g^{-1}(y) \not\subset \mathcal{I}$. Corollary 26 addresses a mixed case using the notion of generalized window perturbations.

Corollary 26. *Let $g \in \mathbb{G}$. Let interval $\mathcal{I} \subset [0, 1]$ and $\mathcal{Y} = g(\mathcal{I})$. Suppose that $\mathcal{I} = \bigcup_{i=1}^m \mathcal{I}_i$ where $\{\mathcal{I}_i\}$ are m intervals with mutually disjoint interiors and g is monotone on every \mathcal{I}_i . Suppose that $c \in (\mathcal{Y}^0, \mathcal{Y}^1)$ exists such that $\forall y \in [\mathcal{Y}^0, c], g^{-1}(y) \subset \mathcal{I}$, and $\forall y \in [c, \mathcal{Y}^1], g^{-1}(y) \not\subset \mathcal{I}$. Then $f_1 \in \mathbb{F}, g_1 \in \mathbb{G}$, and an m -fold window perturbation map w_1 and interval \mathcal{J} exists such that $g = g_1(w_1(f_1))$, the graph of g_1 on \mathcal{J} consists of two affine segments connected by a type I breakpoint, and $g_1([0, \mathcal{J}^0]) \simeq g([0, \mathcal{I}^0]), g_1([\mathcal{J}^1, 1]) \simeq g([\mathcal{I}^1, 1])$.*

Proof. Because g is monotone on every \mathcal{I}_i , \mathcal{I}_i can be partitioned into $\mathcal{I}_i = \mathcal{I}_{i,0} \cup \mathcal{I}_{i,1}$ where $g(\mathcal{I}_{i,0}) = [\mathcal{Y}^0, c]$ and $g(\mathcal{I}_{i,1}) = [c, \mathcal{Y}^1]$. Combining Corollary 21 and Lemma 22, $f_1 \in \mathbb{F}, g_2 \in \mathbb{G}$ and interval $\mathcal{J} = \bigcup_{i=1}^m (\mathcal{J}_{i,0} \cup \mathcal{J}_{i,1})$, where $\{\mathcal{J}_{i,0}, \mathcal{J}_{i,1}\}$ are intervals with mutually disjoint interiors, exist such that $g_2(f_1) = g$, and $g_2([0, \mathcal{J}^0]) \simeq g([0, \mathcal{I}^0]), g_2([\mathcal{J}^1, 1]) \simeq g([\mathcal{I}^1, 1])$. For $i = 1, 2, \dots, m$, the graph of g_2 on $\mathcal{J}_{i,1}$ is an affine segment whose slope has the absolute value $2^{k''_i}$ with $\sum_{i=1}^m 2^{-k''_i} = 2^{-K''}$ for some positive integer K'' , and the graph of g_2 on $\mathcal{J}_{i,0}$ is an affine segment with the absolute value of the slope equal to $2^{k''_i - K''}$. Recall that $\sum_{i=1}^m 2^{-k''_i + K''} = 1$. Hence, g_2 is a generalized m -fold window perturbation on \mathcal{J} where $\mathcal{J}_{i,0} \cup \mathcal{J}_{i,1}$ is a component interval. By Corollary 25, an m -fold window perturbation w_1 on \mathcal{J} exists such that $g_2 = g_1(w_1)$. Interval \mathcal{J} can be partitioned to two intervals $\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}_1$ with mutually disjoint interiors such that the graph of g_1 on \mathcal{J}_0 is affine with absolute value of the slope equal to 1 and the graph of g_1 on \mathcal{J}_1 is affine with absolute value of the slope equal to $2^{K''}$. \square

Corollary 26 holds under a slightly modified hypothesis: $\forall y \in [\mathcal{Y}^0, c], g^{-1}(y) \not\subset \mathcal{I}$, and $\forall y \in (c, \mathcal{Y}^1], g^{-1}(y) \subset \mathcal{I}$. One can further extend the result to a scenario where $c_1, c_2 \in (\mathcal{Y}^0, \mathcal{Y}^1)$ exist with $c_1 < c_2$ such that $\forall y \in [\mathcal{Y}^0, c_1) \cup (c_2, \mathcal{Y}^1], g^{-1}(y) \subset \mathcal{I}$, and $\forall y \in [c_1, c_2], g^{-1}(y) \not\subset \mathcal{I}$. The same conclusion as in Corollary 26 holds except that g_1 on \mathcal{I} consists of three affine segments connected by two type I breakpoints.

In Lemma 22 and Lemma 24 g is required to be an m -fold window perturbation on \mathcal{I} . This requirement is relaxed in Lemma 27.

Lemma 27. *Replacing “ $g \in \mathbb{G}$ is an affine segment on interval \mathcal{I}_{2i} ” in Lemma 22 and replacing “ g is an m -fold window perturbation on interval \mathcal{I} ” in Lemma 24 by “let g be monotone on interval \mathcal{I}_{2i} for all i ”, the conclusions in Lemma 22 and in Lemma 24 still hold.*

Proof. The difference from Lemma 22 and Lemma 24 is that g is not necessarily an affine segment on \mathcal{I}_{2i} . As in the proof of Lemma 22, partition interval \mathcal{Y} into intervals $\{\mathcal{Y}_j\}$, $j = 1, 2, \dots, n$ such that no breakpoint exists whose y -coordinate falls in the interior of any \mathcal{Y}_j . Consider $g^{-1}(\mathcal{Y}_j)$. Let $g^{-1}(\mathcal{Y}_j) = \{\mathcal{I}_{j,1}, \mathcal{I}_{j,2}, \dots, \mathcal{I}_{j,m}, \mathcal{I}_{j,m+1}, \dots, \mathcal{I}_{j,m+n_j}\}$. By the hypothesis of the lemma, interval $\mathcal{I}_{j,i} \subset \mathcal{I}$ for $i = 1, \dots, m$, and interval $\mathcal{I}_{j,i} \cap \mathcal{I} = \emptyset$ for $i = m + 1, \dots, m + n_j$ and $n_j \geq 1$. Let $2^{k_{j,i}}$ be the absolute value of the slope of the affine segment on $\mathcal{I}_{j,i}$.

Let $j^* = \arg \min_{j=1,2,\dots,n} n_j$. If $\arg \min_{j=1,2,\dots,n} n_j$ is not unique, then pick any one of them as j^* . For $j = 1, \dots, n$, let

$$k''_{j,i} = \begin{cases} k_{j^*,i}, & i = 1, 2, \dots, m + n_{j^*} - 1; \\ k_{j^*,m+n_{j^*}} + i - m - n_{j^*} + 1, & i = m + n_{j^*}, m + n_{j^*} + 1, \dots, m + n_j - 1; \\ k_{j^*,m+n_{j^*}} + n_j - n_{j^*}, & i = m + n_j. \end{cases} \quad (23)$$

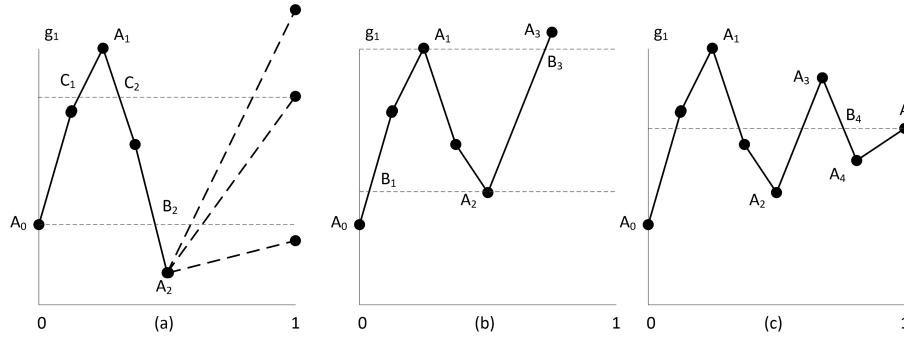


Figure 18: Proof of Theorem 28. In (c), $n = 5$.

It is easy to verify that (22) holds. As in the proof of Lemma 22, one can find $f_1 \in \mathbb{F}, g_1 \in \mathbb{G}$ and a partition of $[0, 1]$ $\{\mathcal{J}_1 < \dots < \mathcal{J}_{2m+1}\}$, where \mathcal{J}_{2i} is further partitioned to $\mathcal{J}_{2i} = \bigcup_{j=1}^n \mathcal{J}_{j,i}$, such that $g_1(f_1(x)) = g(x)$, $g_1(\mathcal{J}_{j,i}) \simeq g(\mathcal{I}_{j,i})$, and the absolute value of the slope changes from $2^{k_{j,i}}$ of g on $\mathcal{I}_{j,i}$ to $2^{k''_{j,i}}$ of g_1 on $\mathcal{J}_{j,i}$ for $i = 1, \dots, n_j$ and $j = 1, \dots, n$, and remains unchanged from g on $[0, 1] \setminus \bigcup_{i=1}^m \mathcal{I}_{2i}$ to g_1 on $[0, 1] \setminus \bigcup_{i=1}^m \mathcal{J}_{2i}$.

Note from the preceding construction (23) that $k''_{j,i}$ does not depend on j when $i = 1, 2, \dots, m$, because $n_{j^*} \geq 1$. Thus, the graph of g_1 on \mathcal{J}_{2i} is one affine segment and Lemma 22 is applicable to g_1 and the conclusion in Lemma 22 still holds.

Moreover, if $\bigcup_{i=1}^m \mathcal{I}_{2i}$ is an interval, then as in the proof of Lemma 24, $\bigcup_{i=1}^m \mathcal{J}_{2i}$ is an interval too, denoted by \mathcal{J} . The graph of g_1 on \mathcal{J} is an m -fold window perturbation. Hence, Lemma 24 is applicable to g_1 and the conclusion in Lemma 24 still holds. \square

In summary, Lemma 19 to Lemma 27 can be used to eliminate type I and type II breakpoints. The following theorem shows that for any $g \in \mathbb{G}$, all interior breakpoints can be eliminated by repetitively applying these lemmas and corollaries. The only \mathbb{G} maps that have no interior breakpoints are the trivial maps $g_{0,+}$ and $g_{0,-}$.

Theorem 28. *Let $g \in \mathbb{G}$. Then g is equal to the composition of a trivial map followed by a combination of \mathbb{F} maps and window perturbations.*

Proof. Suppose that Lemma 19 to Lemma 27 have been applied to eliminate the breakpoints of g such that $g = g_1 \circ f_1 \circ w_1 \circ f_2 \circ w_2 \circ \dots$ where $g_1 \in \mathbb{G}$ and $f_1 \circ w_1 \circ f_2 \circ w_2 \circ \dots$ represent a combination of \mathbb{F} maps and window perturbations. Assume that no more interior breakpoints in g_1 can be eliminated using the preceding lemmas. Next we show that g_1 has no interior breakpoints.

Denote by A_0 the point of g_1 at $A_{0,x} = 0$. Without loss of generality, suppose that the derivative at A_0 is positive. As x increases from 0, $g_1(x)$ increases until it reaches another point A_1 where g_1 stops increasing. If $A_{1,x} = 1$, then $g_1 = g_{0,+}$ and the proof is done. Otherwise, A_1 must be a type II breakpoint and the right derivative is negative. As x increases from $A_{1,x}$, $g_1(x)$ decreases until it reaches another type II breakpoint A_2 .

First, suppose that $A_{2,y} \leq A_{0,y}$. Then a unique point B_2 exists where $A_{1,x} < B_{2,x} < A_{2,x}$ and $B_{2,y} = A_{0,y}$, as shown in Figure 18(a). Consider the following three cases illustrated respectively by three dash lines coming out of point A_2 in Figure 18(a).

- Assume $\max(g_1([A_{2,x}, 1])) < A_{0,y}$. Then $\forall y \in [A_{0,y}, A_{1,y}]$, $\nexists x \in (B_{2,x}, 1]$ such that $g_1(x) = y$. That is, $g_1^{-1}(y)$ consists of two points x_1, x_2 where $x_1 \in [A_{0,x}, A_{1,x}]$, $x_2 \in [A_{1,x}, B_{2,x}]$. Thus, A_0A_1 and A_1B_2 must be affine segments with slopes 2, -2 respectively. By Lemma 23, breakpoint A_1 can be eliminated with a 2-fold window perturbation. Contradiction.

- Assume $A_{0,y} \leq \max(g_1([A_{2,x}, 1])) \leq A_{1,y}$. Let C_1, C_2 be points between A_0, A_1 and between A_1, B_2 respectively such that $C_{1,y} = C_{2,y} = \max(g_1([A_{2,x}, 1]))$. Then $\forall y \in (\max(g_1([A_{2,x}, 1])), A_{1,y}]$, $\nexists x \in (B_{2,x}, 1]$ such that $g_1(x) = y$. Thus, C_1A_1 and A_1C_2 must be affine segments with slopes $2, -2$ respectively. On the other hand, $\forall y \in [A_{0,y}, \max(g_1([A_{2,x}, 1]))]$, $\exists x \in (B_{2,x}, 1]$ such that $g_1(x) = y$. By Lemma 22, one can eliminate all type I breakpoints, if any, between A_0, C_1 and between C_2, B_2 . By Lemma 19, make A_0C_1 and C_2B_2 affine segments with the same slope except for the sign. Therefore, the graph of g_1 on $A_0C_1A_1C_2B_2$ is a generalized 2-fold window perturbation. By Corollary 25, breakpoint A_1 can be eliminated with a 2-fold window perturbation. Contradiction.
- Assume $\max(g_1([A_{2,x}, 1])) > A_{1,y}$. By Lemma 22, one can eliminate all type I breakpoints, if any, between A_0, A_1 and between A_1, B_2 . By Lemma 19, make A_0A_1 and A_1B_2 affine segments with the same slope except for the sign. Therefore, the graph of g_1 on $A_0A_1B_2$ is a 2-fold window perturbation. By Lemma 23, breakpoint A_1 can be eliminated with a 2-fold window perturbation. Contradiction.

In the following, suppose that $A_{2,y} > A_{0,y}$. Then a unique point B_1 exists where $A_{0,x} < B_{1,x} < A_{1,x}$ and $B_{1,y} = A_{2,y}$. As x increases from $A_{2,x}$, $g_1(x)$ increases until it reaches another type II breakpoint A_3 . If $A_{3,y} \geq A_{1,y}$, then a unique point B_3 exists where $A_{2,x} < B_{3,x} < A_{3,x}$ and $B_{3,y} = A_{1,y}$, as shown in Figure 18(b). By Corollary 26 and Lemma 27, type II breakpoints A_1 and A_2 can both be eliminated. Contradiction.

Therefore, $A_{3,y} < A_{1,y}$. The process continues as shown in Figure 18(c). For odd i , type II breakpoint A_i is facing down and $A_{i,y} > A_{i+2,y}$. For even i , type II breakpoint A_i is facing up and $A_{i,y} < A_{i+2,y}$. $A_{2i+1,y} > A_{2j,y}$ for any i, j . Suppose that A_n is the endpoint where $A_{n,x} = 1$. $\min(A_{n-1,y}, A_{n-2,y}) < A_{n,y} < \max(A_{n-1,y}, A_{n-2,y})$. Therefore, a unique point B_{n-1} exists where $A_{n-2,x} < B_{n-1,x} < A_{n-1,x}$ and $B_{n-1,y} = A_{n,y}$. By Lemma 22, one can eliminate all type I breakpoints, if any, between B_{n-1}, A_{n-1} and between A_{n-1}, A_n . By Lemma 19, make $B_{n-1}A_{n-1}$ and A_{n-1}, A_n affine segments with the same slope except for the sign. Therefore, the graph of g_1 on $B_{n-1}A_{n-1}A_n$ is a 2-fold window perturbation. By Lemma 23, breakpoint A_{n-1} can be eliminated with a 2-fold window perturbation. Contradiction.

Hence, we conclude that g_1 has no interior breakpoints. □

Because any \mathbb{F} map can be generated by the two generators defined in (1), it suffices to study the basic maps in \mathbb{G} to generate the window perturbations thanks to Theorem 28.

Denote by $w_{m,\mathcal{J}}$ an m -fold window perturbation map where $w_{m,\mathcal{J}}(x) = x$ for $x \in [0, 1] \setminus \mathcal{J}$ and $w_{m,\mathcal{J}}(x)$ is an m -fold window perturbation on \mathcal{J} . Specifically, let $\{\mathcal{J}_1 < \mathcal{J}_2 < \dots < \mathcal{J}_m\}$ be a partition of \mathcal{J} . The graph of $w_{m,\mathcal{J}}$ is an affine segment on each \mathcal{J}_i with slope $(-1)^{i-1}2^{k_i}$ where $\sum_{i=1}^m 2^{-k_i} = 1$. The graph of $w_{m,\mathcal{J}}$ on \mathcal{J}_i is referred to as the i -th leg of $w_{m,\mathcal{J}}$. Map $w_{m,\mathcal{J}}$ defined here is from the lower left corner to the upper right corner. Map $1 - w_{m,\mathcal{J}}$, which is from the upper left corner to the lower right corner, can be generated by $g_{0,-}(w_{m,\mathcal{J}})$.

Lemma 29. Any $(m + 2)$ -fold window perturbation map $w_{m+2,\mathcal{J}}$ on interval \mathcal{J} is equal to $w_{m,\mathcal{J}}(w(f))$ where w is a 3-fold window perturbation w and $f \in \mathbb{F}$.

Proof. Let $w_{m,\mathcal{J}}$ be a m -fold window perturbation. Let \mathcal{J}_m be the interval corresponding to the m -th leg of $w_{m,\mathcal{J}}$. Let w_{3,\mathcal{J}_m} be a 3-fold window perturbation on \mathcal{J}_m with the absolute values of the slopes being 2^{q_j} for $j = 1, 2, 3$ on the three legs respectively. By definition, $\sum_{j=1}^3 2^{-q_j} = 1$.

By construction, $w_{m,\mathcal{J}}(w_{3,\mathcal{J}_m})$ is an $(m + 2)$ -fold window perturbation map on interval \mathcal{J} where $w_{m,\mathcal{J}}(w_{3,\mathcal{J}_m})(x) = w_{m,\mathcal{J}}(x)$ for $x \in [0, 1] \setminus \mathcal{J}_m$ and $w_{m,\mathcal{J}}(w_{3,\mathcal{J}_m})(x)$ consists of three legs on \mathcal{J}_m whose slopes are $(-1)^{m-1+j}2^{k_m+q_j}$ for $j = 1, 2, 3$. Interval \mathcal{J}_m is thus partitioned to three intervals \mathcal{J}_{m_j} corresponding to the three legs.

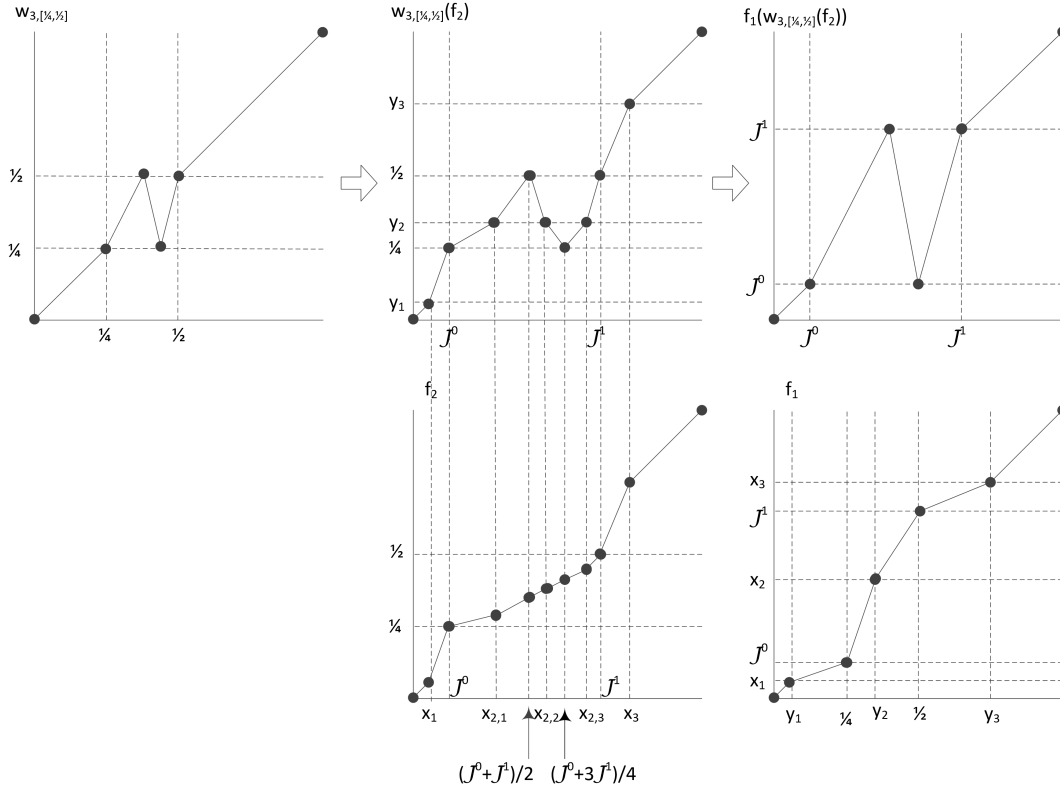


Figure 19: Construction of $w_{3,\mathcal{J}}$ with $\bar{w}_{3, [\frac{1}{4}, \frac{1}{2}]}$.

Let $\mathcal{J}'_1 < \mathcal{J}'_2 < \dots < \mathcal{J}'_{m+2}$ be the partition of \mathcal{J} of any desired $(m + 2)$ -fold window perturbation $w_{m+2,\mathcal{J}}$, which is an affine segment on each \mathcal{J}'_i with slope $(-1)^{i-1}2^{l_i}$. Because

$$\sum_{i=1}^{m+2} 2^{-l_i} = \sum_{i=1}^{m-1} 2^{-k_i} + \sum_{j=1}^3 2^{-k_m - q_j} = 1,$$

by Lemma 19, $f \in \mathbb{F}$ exists to map \mathcal{J}'_i to \mathcal{J}_i for $i = 1, \dots, m - 1$ and \mathcal{J}'_{m-1+j} to \mathcal{J}_m for $j = 1, 2, 3$ without altering anything on $[0, 1] \setminus \mathcal{J}$. Hence, $w_{m+2,\mathcal{J}} = w_{m,\mathcal{J}}(w_{3,\mathcal{J}_m}(f))$. \square

From Lemma 29, any m -fold window perturbation $w_{m,\mathcal{J}}$ can be generated by repetitively applying 3-fold window perturbations on appropriate intervals of \mathcal{J} to 1-fold $w_{1,\mathcal{J}}$ for odd m or 2-fold window perturbation $w_{2,\mathcal{J}}$ for even m . The 1-fold window perturbation $w_{1,\mathcal{J}}$ is simply $g_{0,+}$ or $g_{0,-}$. Next we show that all 3-fold or 2-fold window perturbations can be generated by a finite number of basic window perturbations.

Define the basic 3-fold window perturbation $\bar{w}_{3, [\frac{1}{4}, \frac{1}{2}]}$ as the special case of $w_{3, [\frac{1}{4}, \frac{1}{2}]}$ with the absolute values of the slopes being 2, 4, 4 on the three legs respectively. Lemma 30 states that almost any 3-fold window perturbation can be generated by the the basic 3-fold window perturbation. The remaining cases of 3-fold window perturbations are addressed in Lemma 32.

Lemma 30. Any 3-fold window perturbation $w_{3,\mathcal{J}}$ is equal to $f_1(\bar{w}_{3, [\frac{1}{4}, \frac{1}{2}]}(f_2))$ for $f_1, f_2 \in \mathbb{F}$ if $0 < \mathcal{J}^0 < \mathcal{J}^1 < 1$.

Proof. We prove the lemma by construction as illustrated in Figure 19.

Map f_2 , which scales $\bar{w}_{3, [\frac{1}{4}, \frac{1}{2}]}$ horizontally to $\bar{w}_{3, [\frac{1}{4}, \frac{1}{2}]}(f_2)$, does the following.

- (a) Map $[0, \mathcal{J}^0]$ to $[0, \frac{1}{4}]$. If $\frac{1}{\mathcal{J}^0}$ is in the form of 2^k , then f_2 on $[0, \mathcal{J}^0]$ is an affine segment; otherwise, f_2 on $[0, \mathcal{J}^0]$ consists of two affine segments separated by point (x_1, y_1) , a partition point between points $(0, 0)$ and $(\mathcal{J}^0, \frac{1}{4})$ by Lemma 3.
- (b) Map $[\mathcal{J}^0, \mathcal{J}^1]$ to $[\frac{1}{4}, \frac{1}{2}]$. If $\frac{\frac{1}{2}-\frac{1}{4}}{\mathcal{J}^1-\mathcal{J}^0}$ is in the form of 2^k , then f_2 on $[\mathcal{J}^0, \mathcal{J}^1]$ is an affine segment; otherwise, a point (x_2, y_2) exists in Lemma 3 such that $\frac{\frac{1}{2}-y_2}{\mathcal{J}^1-x_2}$ and $\frac{y_2-\frac{1}{4}}{x_2-\mathcal{J}^0}$ are both in the form of 2^k . The graph of f_2 on $[\mathcal{J}^0, \mathcal{J}^1]$ consists of six affine segments separated by partition points $(x_{2,1}, \frac{1}{4} + \frac{y_2-\frac{1}{4}}{2})$, $(\frac{\mathcal{J}^0+\mathcal{J}^1}{2}, \frac{3}{8})$, $(x_{2,2}, \frac{3}{8} + \frac{\frac{1}{2}-y_2}{4})$, $(\frac{\mathcal{J}^0+3\mathcal{J}^1}{4}, \frac{7}{16})$ and $(x_{2,3}, \frac{7}{16} + \frac{y_2-\frac{1}{4}}{4})$ between points $(\mathcal{J}^0, \frac{1}{4})$ and $(\mathcal{J}^1, \frac{1}{2})$, where $x_{2,1} = \mathcal{J}^0 + \frac{x_2-\mathcal{J}^0}{2}$, $x_{2,2} = \frac{\mathcal{J}^0+\mathcal{J}^1}{2} + \frac{\mathcal{J}^1-x_2}{4}$ and $x_{2,3} = \frac{\mathcal{J}^0+3\mathcal{J}^1}{4} + \frac{x_2-\mathcal{J}^0}{4}$. Note that the derivative of f_2 is $\frac{y_2-\frac{1}{4}}{x_2-\mathcal{J}^0}$ on $[\mathcal{J}^0, x_{2,1}]$, $[x_{2,2}, \frac{\mathcal{J}^0+3\mathcal{J}^1}{4}]$ and $[\frac{\mathcal{J}^0+3\mathcal{J}^1}{4}, x_{2,3}]$, and is $\frac{\frac{1}{2}-y_2}{\mathcal{J}^1-x_2}$ on $[x_{2,1}, \frac{\mathcal{J}^0+\mathcal{J}^1}{2}]$, $[\frac{\mathcal{J}^0+\mathcal{J}^1}{2}, x_{2,2}]$ and $[x_{2,3}, \mathcal{J}^1]$.
- (c) Map $[\mathcal{J}^1, 1]$ to $[\frac{1}{2}, 1]$. If $\frac{1-\frac{1}{2}}{1-\mathcal{J}^1}$ is in the form of 2^k , then f_2 on $[\mathcal{J}^1, 1]$ is an affine segment; otherwise, f_2 on $[\mathcal{J}^1, 1]$ consists of two affine segments separated by point (x_3, y_3) , a partition point between points $(\mathcal{J}^1, \frac{1}{2})$ and $(1, 1)$ by Lemma 3.

Map f_1 , which scales $\bar{w}_{3, [\frac{1}{4}, \frac{1}{2}]}(f_2)$ vertically to $f_1(\bar{w}_{3, [\frac{1}{4}, \frac{1}{2}]}(f_2))$, does the following.

- (a) Map $[0, \frac{1}{4}]$ to $[0, \mathcal{J}^0]$. If $\frac{\mathcal{J}^0}{\frac{1}{4}}$ is in the form of 2^k , then f_1 on $[0, \frac{1}{4}]$ is an affine segment; otherwise, by symmetry, f_1 on $[0, \frac{1}{4}]$ consists of two affine segments separated by point (y_1, x_1) , where x_1 and y_1 are given in the preceding step (a).
- (b) Map $[\frac{1}{4}, \frac{1}{2}]$ to $[\mathcal{J}^0, \mathcal{J}^1]$. If $\frac{\mathcal{J}^1-\mathcal{J}^0}{\frac{1}{2}-\frac{1}{4}}$ is in the form of 2^k , then f_1 on $[\frac{1}{4}, \frac{1}{2}]$ is an affine segment; otherwise, by symmetry, f_1 on $[\frac{1}{4}, \frac{1}{2}]$ consists of two affine segments separated by point (y_2, x_2) , where x_2 and y_2 are given in the preceding step (b).
- (c) Map $[\frac{1}{2}, 1]$ to $[\mathcal{J}^1, 1]$. If $\frac{1-\frac{1}{2}}{1-\mathcal{J}^1}$ is in the form of 2^k , then f_1 on $[\frac{1}{2}, 1]$ is an affine segment; otherwise, by symmetry, f_1 on $[\frac{1}{2}, 1]$ consists of two affine segments separated by (y_3, x_3) , where x_3 and y_3 are given in the preceding step (c).

Hence, f_1 and f_2 scale and translate the 3-fold window perturbation on $[\frac{1}{4}, \frac{1}{2}]$ to $[\mathcal{J}^0, \mathcal{J}^1]$. Finally, the absolute values of the slopes are $2, 2, 4$ on the three legs of $f_1(\bar{w}_{3, [\frac{1}{4}, \frac{1}{2}]}(f_2))$. One can apply $f_3 \in \mathbb{F}$ such that $f_1(\bar{w}_{3, [\frac{1}{4}, \frac{1}{2}]}(f_2(f_3)))$ achieves any $2^{q_1}, 2^{q_2}, 2^{q_3}$ by Lemma 19. \square

Next consider 2-fold window perturbations.

Lemma 31. Any 2-fold window perturbation $w_{2, \mathcal{J}}$ can be generated by $w_{2, [\frac{3}{4}, 1]}$ if $\mathcal{J} \subset [0, 1]$.

Proof. As in Lemma 30, it can be shown with scaling construction that any 2-fold window perturbation $w_{2, \mathcal{J}}$ can be generated with $f_1(w_{2, [\frac{3}{4}, 1]}(f_2))$ for $f_1, f_2 \in \mathbb{F}$ if $0 < \mathcal{J}^0 < \mathcal{J}^1 = 1$. On the other hand, w_{2, \mathcal{J}^1} with $0 = \mathcal{J}_1^0 < \mathcal{J}_1^1 < 1$ can be generated with $g_{0, -}(w_{2, \mathcal{J}}(g_{0, -}))$ where $\mathcal{J} = [1 - \mathcal{J}_1^1, 1]$. Hence, the conclusion follows. \square

Finally consider the special cases of 3-fold window perturbations that are not addressed in Lemma 30.

Lemma 32. Any 3-fold window perturbation $w_{3, \mathcal{J}}$ with $\mathcal{J}^0 = 0$ and/or $\mathcal{J}^1 = 1$ is equal to the composition of 2-fold window perturbations and $f \in \mathbb{F}$.

Proof. Suppose $\mathcal{J}^1 = 1$. The case of $\mathcal{J}^0 = 0$ can be addressed similarly.

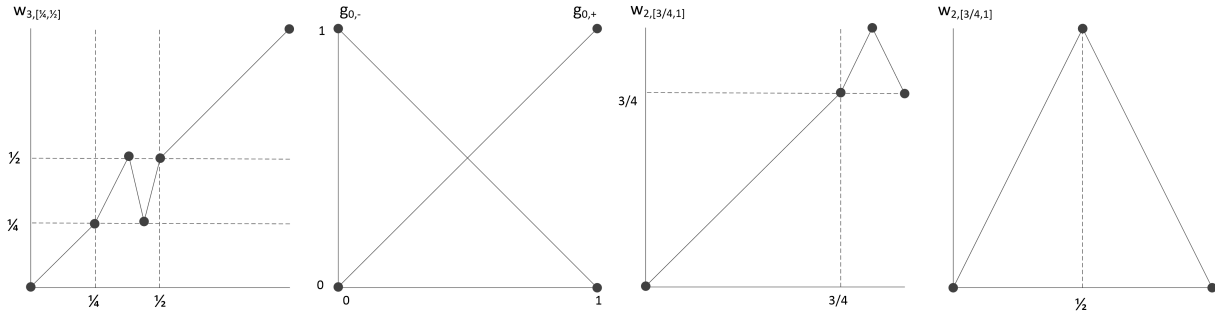


Figure 20: Plots of basic maps in \mathbb{G} : $\bar{w}_{3,[\frac{1}{4},\frac{1}{2}]}$, $g_{0,+}$, $g_{0,-}$, $w_{2,[\frac{3}{4},1]}$, and $w_{2,[0,1]}$

Consider the 3-fold window perturbation $\bar{w}_{3,\mathcal{J}}$ as the special case of $w_{3,\mathcal{J}}$ with the absolute values of the slopes being 2, 4, 4 on the three legs respectively.

$$\bar{w}_{3,\mathcal{J}} = w_{2,\mathcal{J}} \left(w_{2,\mathcal{J}_1} \right),$$

where interval $\mathcal{J}_1 = [\frac{\mathcal{J}^0+1}{2}, 1]$. Then any $w_{3,\mathcal{J}} = \bar{w}_{3,\mathcal{J}}(f)$ for some $f \in \mathbb{F}$ by Lemma 19. □

The follow theorem follows from Theorem 28 and Lemmas 29, 30, 31 and 32.

Theorem 33. *Let $g \in \mathbb{G}$. Then g is equal to the composition of a combination of $g_{0,+}$, $g_{0,-}$, $\bar{w}_{3,[\frac{1}{4},\frac{1}{2}]}$, $w_{2,[\frac{3}{4},1]}$, $w_{2,[0,1]}$ and the two generator maps of \mathbb{F} defined in (1).*

Figure 20 plots $g_{0,-}$, $\bar{w}_{3,[\frac{1}{4},\frac{1}{2}]}$, $w_{2,[\frac{3}{4},1]}$, and $w_{2,[0,1]}$.

Finally, Theorem 35 shows that unlike \mathbb{F} , \mathbb{G} is not finitely generated. To this end, Lemma 34 studies the number of type II breakpoints of a composition map in \mathbb{G} . Denote by $\#(g)$ the number of type II breakpoints of map g .

Lemma 34. *Let $g_1, g_2 \in \mathbb{G}$. Then $\#(g_1 \circ g_2) \geq \#(g_1) + \#(g_2)$.*

Proof. Consider two cases.

Case 1. Suppose that point B on the graph of g_2 is a type II breakpoint. There exists $\delta > 0$ such that the graph of g_2 is an affine segment on $[B_x - \delta, B_x]$ and a different affine segment on $[B_x, B_x + \delta]$ and the slopes of the two affine segments are of different signs. Either $g_2(B_x - \delta) > g_2(B_x)$ and $g_2(B_x + \delta) > g_2(B_x)$, or $g_2(B_x - \delta) < g_2(B_x)$ and $g_2(B_x + \delta) < g_2(B_x)$. Thus a sufficiently small δ exists such that the graph of g_1 is an affine segment on both $\langle g_2(B_x - \delta), g_2(B_x) \rangle$ and $\langle g_2(B_x + \delta), g_2(B_x) \rangle$. Therefore, $g_1(g_2(B_x))$ is a type II breakpoint on the graph of $g_1 \circ g_2$. That is, every type II breakpoint of g_2 corresponds to at least one type II breakpoint of $g_1 \circ g_2$.

Case 2. Suppose that point A on the graph of g_1 is a type II breakpoint. Because g_2 is a continuous map onto $[0, 1]$, point C exists on the graph of g_2 such that $C_x \in g_2^{-1}(A_x)$ and $\delta > 0$ exists such that the graph of g_2 on $[C_x - \delta, C_x + \delta]$ is monotone. Following the preceding argument in case 1, $g_1(g_2(C_x))$ is a type II breakpoint on the graph of $g_1 \circ g_2$. That is, every type II breakpoint of g_1 corresponds to at least one type II breakpoint of $g_1 \circ g_2$. Because point C is not a type II breakpoint of g_2 , this type II breakpoint on the graph of $g_1 \circ g_2$ is not included in the preceding case 1 and there is no double counting between cases 1 and 2. Hence, $\#(g_1 \circ g_2) \geq \#(g_1) + \#(g_2)$. □

Theorem 35. *\mathbb{G} is not finitely generated.*

Proof. For any dyadic number $\delta \in (0, 1)$, $\#(w_{2,[\delta,1]}) = 1$. If $g_1, g_2 \in \mathbb{G}$ exist such that $w_{2,[\delta,1]} = g_1 \circ g_2$, then by Lemma 34, one of g_1 and g_2 is a trivial map. Thus, $w_{2,[\delta,1]}$ cannot be generated by $w_{2,[\delta',1]}$ with

another dyadic number $\delta' \neq \delta$ or other maps in \mathbb{G} with two or more type II breakpoints. As there are infinitely many δ , the set of $\{w_{2, [\delta, 1]}\}$ cannot be generated by a finite number of generators. \square

8 Equivalence Classes and Generators

Comparison of Theorem 33 and Theorem 35 indicates that the maps in \mathbb{F} play an important role in allowing a finite number of basic maps to generate any map in \mathbb{G} . To study this idea formally, define the notion of equivalence relation and equivalence classes.

Definition 14 (Equivalence Relation). A binary relation \sim on \mathbb{G} is defined as follows. Suppose that $g_1, g_2 \in \mathbb{G}$. $g_1 \sim g_2$ if and only if $f_1, f_2 \in \mathbb{F}$ exist such that $g_2 = f_1 \circ g_1 \circ f_2$. Binary relation \sim is an equivalence relation because it is reflexive, symmetric and transitive.

Definition 15 (Equivalence Class). The equivalence class of $g \in \mathbb{G}$, denoted by $[g]$, is the set $\{\hat{g} \in \mathbb{G} | \hat{g} \sim g\}$.

To understand the effect of f_1 and f_2 on g in Definition 15, let $g \in \mathbb{G}$ and $f \in \mathbb{F}$. Suppose that the graph of f is an affine segment on interval $\mathcal{I}_0 = f^{-1}(\mathcal{I}_1)$. The slope of the affine segment is $s = \frac{|\mathcal{I}_1|}{|\mathcal{I}_0|}$. From the properties of \mathbb{F} , a portion of the graph of g is scaled at a ratio of s to become part of $g \circ f$ or $f \circ g$. Specifically, to obtain $g \circ f$, the graph of g on \mathcal{I}_1 is scaled *horizontally* to an interval \mathcal{I}'_0 with $|\mathcal{I}'_0| = |\mathcal{I}_0|$. The exact location of \mathcal{I}'_0 on $[0, 1]$ is such that the continuity is maintained in $g \circ f$ and thus depends on the scaling of other portions. To obtain $f \circ g$, the graph of g on $g^{-1}(\mathcal{I}_0)$ is scaled *vertically* to an interval \mathcal{I}'_1 with $|\mathcal{I}'_1| = |\mathcal{I}_1|$. The exact location of \mathcal{I}'_1 on $[0, 1]$ is such that the continuity is maintained in $f \circ g$. Figure 21 illustrates the scaling operation of an affine segment of f .

Lemma 36. Let $f_1 \in \mathbb{F}$ and $g \in \mathbb{G}$. Then there exists $f_2 \in \mathbb{F}$ such that $f_1 \circ g \circ f_2 \in \mathbb{G}$.

Proof. Partition $[0, 1]$ into a set of intervals $\{\mathcal{Y}_i\}$ such that the interiors of $f_1^{-1}(\mathcal{Y}_i)$ contains no breakpoints of f_1 and the endpoints of $\{\mathcal{Y}_i\}$ are all dyadic for all i . Suppose that the derivative of f_1 on $f_1^{-1}(\mathcal{Y}_i)$ is s_i . From g to $f_1 \circ g$, f_1 vertically scales the graph of g on $g^{-1}(f_1^{-1}(\mathcal{Y}_i))$ by a factor of s_i .

In the rectangle diagram representation of a map $f \in \mathbb{F}$, $[0, 1]$ is partitioned into $\{\mathcal{I}_i\}$ and $f \in \mathbb{F}$ is completely defined by specifying a scaling factor from \mathcal{I}_i to \mathcal{J}_i , for $i = 1, 2, \dots$, where $\{\mathcal{J}_i\}$ is another interval partition of $[0, 1]$. To construct f_2 , let the scaling factor be s_i on $g^{-1}(f_1^{-1}(\mathcal{Y}_i))$ for all i . Let $g^{-1}(f_1^{-1}(\mathcal{Y}_i)) = \bigcup_j \mathcal{I}_{i,j}$ where $\mathcal{I}_{i,1}, \mathcal{I}_{i,2}, \dots$ are intervals of mutually disjoint interiors. Map f_2 scales the graph of $f_1 \circ g$ on $\mathcal{I}_{i,j}$ horizontally to the graph of $f_1 \circ g \circ f_2$ on $\mathcal{J}_{i,j}$ by s_i and thus $|\mathcal{J}_{i,j}| = s_i |\mathcal{I}_{i,j}|$. Because

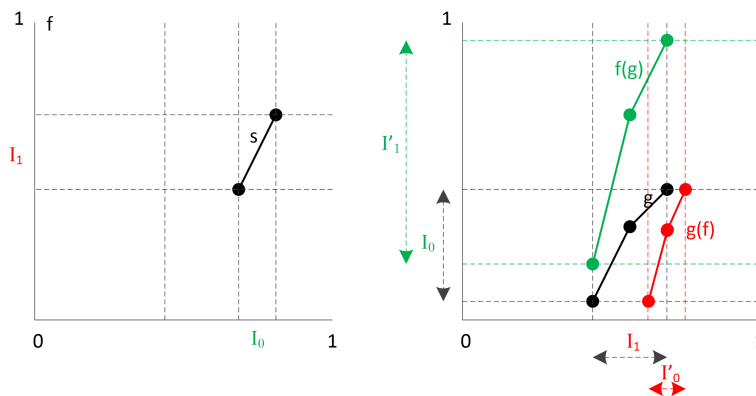


Figure 21: The scaling operation of $g \circ f$ and $f \circ g$ on the graph of g by an affine segment of f .

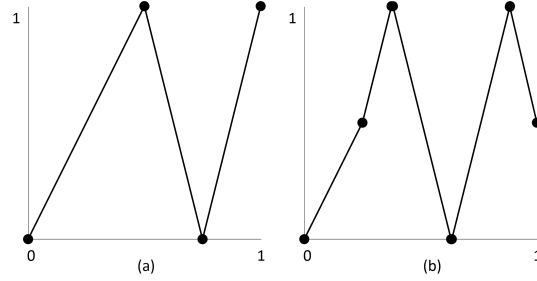


Figure 22: A counterexample to show that (a) $\hat{g}_{1,1} \circ \hat{g}_2$ and (b) $\hat{g}_{1,2} \circ \hat{g}_2$ are not in the same equivalence class. $\hat{g}_{1,1}, \hat{g}_{1,2} \in [g_1]$ where $g_1 = w_{2, [\frac{3}{4}, 1]}$ and $\hat{g}_2 = w_{2, [0, 1]}$.

g is λ -preserving,

$$\sum_j |\mathcal{I}_{i,j}| = |f_1^{-1}(\mathcal{Y}_i)| = \frac{|\mathcal{Y}_i|}{s_i} \Rightarrow \sum_i \sum_j |\mathcal{J}_{i,j}| = \sum_i |\mathcal{Y}_i| = 1.$$

Therefore, $\mathcal{J}_{i,j}$ is a valid partition of $[0, 1]$. Moreover, s_i is in the form of 2^k for integer k and the endpoints of $g^{-1}(f_1^{-1}(\mathcal{Y}_i))$ are dyadic because $f_1 \in \mathbb{F}$ and $g \in \mathbb{G}$. Therefore, $f_2 \in \mathbb{F}$.

From $f_1 \circ g$ to $f_1 \circ g \circ f_2$, f_2 horizontally scales the graph of $f_1 \circ g$ on $g^{-1}(f_1^{-1}(\mathcal{Y}_i))$ by a factor of s_i . Combining the two steps, from g to $f_1 \circ g \circ f_2$, the graph of g on $g^{-1}(f_1^{-1}(\mathcal{Y}_i))$ is scaled horizontally and vertically by the same factor for all i . Because $g \in \mathbb{G}$, it follows that $f_1 \circ g \circ f_2 \in \mathbb{G}$. \square

Lemma 37. *The equivalence classes defined in Definition 15 form a partition of set \mathbb{G} .*

Proof. Because any map in \mathbb{F} is invertible, it follows that if $\hat{g} \in [g]$, then $g \in [\hat{g}]$ and $[g] = [\hat{g}]$, and if $\hat{g}_1, \hat{g}_2 \in [g]$, then $[\hat{g}_1] = [\hat{g}_2]$. Therefore, any map in \mathbb{G} is in exactly one equivalence class. \square

However, the equivalence classes do not form a monoid. Consider the following definition of a binary operation \odot on $[g]$. Let $[g] = [g_1] \odot [g_2]$ where $\hat{g} \in [g]$ if and only if $\hat{g}_1 \in [g_1]$ and $\hat{g}_2 \in [g_2]$ exist such that $\hat{g} \in [\hat{g}_1 \circ \hat{g}_2]$. Example 3 shows that $[g_1] \odot [g_2]$ is not necessarily a single equivalence class.

Example 3. Let $g_1 = w_{2, [\frac{3}{4}, 1]}$ and $g_2 = w_{2, [0, 1]}$. Let $\hat{g}_2 = w_{2, [0, 1]} \in [g_2]$. Consider two elements in equivalence class $[g_1]$: $\hat{g}_{1,1} = w_{2, [\frac{1}{2}, 1]} \in [g_1]$ and $\hat{g}_{1,2} = w_{2, [\frac{1}{4}, 1]} \in [g_1]$. Figure 22 compares $\hat{g}_{1,1} \circ \hat{g}_2$ and $\hat{g}_{1,2} \circ \hat{g}_2$ and shows that they are not in the same equivalence class.

To avoid the technical difficulty of working with equivalence classes directly, consider the notion of sets of equivalence classes instead.

Definition 16 (Set of Equivalence Classes). Let $\Phi \subset \mathbb{G}$. Let $\{[g]_{g \in \Phi}\}$ be the set of equivalence classes $[g]$, $\forall g \in \Phi$. Define a binary operation \odot on $\{[g]_{g \in \Phi}\}$ as follows: $\{[g]_{g \in \Phi_1}\} \odot \{[g]_{g \in \Phi_2}\}$ is the set of equivalence classes $[\hat{g}_1 \circ \hat{g}_2]$ where $g_1 \in \Phi_1$ and $g_2 \in \Phi_2$ exist such that $\hat{g}_1 \in [g_1]$ and $\hat{g}_2 \in [g_2]$.

Lemma 38. *Let $\Phi_1, \Phi_2, \Phi_3 \subset \mathbb{G}$. Then*

$$(\{[g]_{g \in \Phi_1}\} \odot \{[g]_{g \in \Phi_2}\}) \odot \{[g]_{g \in \Phi_3}\} = \{[g]_{g \in \Phi_1}\} \odot (\{[g]_{g \in \Phi_2}\} \odot \{[g]_{g \in \Phi_3}\}).$$

Proof. By Definition 16, if $g \in (\{[g]_{g \in \Phi_1}\} \odot \{[g]_{g \in \Phi_2}\}) \odot \{[g]_{g \in \Phi_3}\}$, then $f_1, f_2, \dots, f_{10} \in \mathbb{F}$ exist such that for some $g_1 \in \Phi_1, g_2 \in \Phi_2, g_3 \in \Phi_3$,

$$g = f_1 \circ \left(\left(f_2 \circ \left((f_3 \circ g_1 \circ f_4) \circ (f_5 \circ g_2 \circ f_6) \right) f_7 \right) \circ (f_8 \circ g_3 \circ f_9) \right) \circ f_{10}.$$

Let

$$g = f'_1 \circ \left((f'_2 \circ g_1 \circ f'_3) \circ \left(f'_4 \circ \left((f'_5 \circ g_2 \circ f'_6) \circ (f'_7 \circ g_3 \circ f'_8) \right) \circ f'_9 \right) \right) \circ f'_{10},$$

where $f'_1, \dots, f'_{10} \in \mathbb{F}$ are determined such that

$$\begin{cases} f_1 \circ f_2 \circ f_3 = f'_1 \circ f'_2, \\ f_4 \circ f_5 = f'_3 \circ f'_4 \circ f'_5, \\ f_6 \circ f_7 \circ f_8 = f'_6 \circ f'_7, \\ f_9 \circ f_{10} = f'_8 \circ f'_9 \circ f'_{10}, \end{cases}$$

and $f'_2 \circ g_1 \circ f'_3, f'_5 \circ g_2 \circ f'_6, f'_7 \circ g_3 \circ f'_8, f'_4 \circ \left((f'_5 \circ g_2 \circ f'_6) \circ (f'_7 \circ g_3 \circ f'_8) \right) \circ f'_9$ are all in \mathbb{G} . Let $f'_1 = g_{0,+}$ and $f'_2 = f_1 \circ f_2 \circ f_3$. From Lemma 36, f'_3 exists to make $f'_2 \circ g_1 \circ f'_3 \in \mathbb{G}$. Next, let $f'_4 = g_{0,+}$ and $f'_5 = (f'_3)^{-1} \circ f_4 \circ f_5$ and f'_6 exists to make $f'_5 \circ g_2 \circ f'_6 \in \mathbb{G}$. Let $f'_7 = (f'_6)^{-1} \circ f_6 \circ f_7 \circ f_8$ and f'_8 exists to make $f'_7 \circ g_3 \circ f'_8 \in \mathbb{G}$. Finally, f'_9 exists such that $f'_4 \circ \left((f'_5 \circ g_2 \circ f'_6) \circ (f'_7 \circ g_3 \circ f'_8) \right) \circ f'_9 \in \mathbb{G}$. Let $f'_{10} = (f'_9)^{-1} \circ (f'_8)^{-1} \circ f_9 \circ f_{10}$. Hence, by Definition 16, $g \in \{[g]_{g \in \Phi_1}\} \circ (\{[g]_{g \in \Phi_2}\} \circ \{[g]_{g \in \Phi_3}\})$. This completes the proof. \square

Theorem 39. *Let $\Phi_a = \{g_{0,+}\}, \Phi_b = \{g_{0,-}\}, \Phi_c = \{\bar{w}_{3, [\frac{1}{4}, \frac{1}{2}]\}, \Phi_d = \{w_{2, [\frac{3}{4}, 1]}\}$ and $\Phi_e = \{w_{2, [0, 1]}\}$. Construct a collection of sets of equivalence classes, each of which is equal to $\{[g]_{g \in \Phi_1}\} \circ \{[g]_{g \in \Phi_2}\} \circ \dots$ where Φ_i for any i is one of $\Phi_a, \Phi_b, \Phi_c, \Phi_d, \Phi_e$. Then, the collection is a monoid and finitely generated. The union of all the elements of the collection is a set of equivalence classes, the union of which is \mathbb{G} .*

Proof. By Lemma 38, associativity holds for the elements in the collection. Equivalence class set $\{[g]_{g \in \Phi_e}\}$ is the identity element. The collection is thus a monoid. By construction, $\{[g]_{g \in \Phi_a}\}, \{[g]_{g \in \Phi_b}\}, \{[g]_{g \in \Phi_c}\}, \{[g]_{g \in \Phi_d}\}$ and $\{[g]_{g \in \Phi_e}\}$ are the generators of the monoid. By Theorem 33, any map $g \in \mathbb{G}$ is equal to the composition of a combination of maps, each of which is in one of equivalence classes $[g]_{g \in \Phi_a}, [g]_{g \in \Phi_b}, [g]_{g \in \Phi_c}, [g]_{g \in \Phi_d}$ and $[g]_{g \in \Phi_e}$. Therefore, the last part of the theorem holds. \square

Next we characterize $[g]$. Partition $[0, 1]$ into intervals $\{\mathcal{Y}_l\}, l = 1, 2, \dots, m$ with $|\mathcal{Y}_l| > 0$ for all l and $\mathcal{Y}_1 < \mathcal{Y}_2 < \dots < \mathcal{Y}_m$ such that no breakpoint exists whose y -coordinate falls in the interior of any \mathcal{Y}_l , i.e., no breakpoint B exists such that $B_y \in (\mathcal{Y}_l^0, \mathcal{Y}_l^1)$ for any l . As x increases from 0 to 1, $g(x)$ moves from one interval to another or stays in one interval but changes the sign of the derivative. We characterize g by the sequence of the indices, referred to as *evolution sequence*, representing the intervals on which $g(x)$ resides.

More precisely, let $\mathcal{I}_1 < \dots < \mathcal{I}_n$ be a partition of $[0, 1]$ such that for any $i = 1, 2, \dots, n$, a unique $l_i \in \{1, 2, \dots, m\}$ exists where $g(\mathcal{I}_i) = \mathcal{Y}_{l_i}$ and the graph of g is affine on every \mathcal{I}_i . Because no breakpoint exists inside any of $\{\mathcal{Y}_l\}$, the construction of $\{\mathcal{I}_i\}$ exists and is unique. The evolution sequence is defined as $\pm l_1 l_2 \dots l_n$, where the sign $+$ or $-$ represents the sign of the derivative of g in interval \mathcal{I}_1 . In an evolution sequence $\pm l_1 l_2 \dots l_n$, adjacent indices l_i and l_{i+1} differ at most by 1. In $\mathcal{I}_i \cup \mathcal{I}_{i+1}$, g is increasing if $l_{i+1} - l_i = 1$ or decreasing if $l_{i+1} - l_i = -1$. If $l_{i+1} - l_i = 0$, g alternates between increasing and decreasing in \mathcal{I}_i and \mathcal{I}_{i+1} .

Suppose that $\hat{g} \in [g]$. From the scaling operation illustrated in Figure 21, partitions $\{\mathcal{I}_i\}$ and $\{\mathcal{I}'_i\}$ of $[0, 1]$ for $i = 1, \dots, n$ exist such that for all i , $\hat{g}(\mathcal{I}'_i) \simeq c_i g(\mathcal{I}_i) + d_i$ for some numbers c_i, d_i . The following lemma follows immediately.

Lemma 40. *If $g_1, g_2 \in \mathbb{G}$ have the same evolution sequence, then g_1 and g_2 are in the same equivalence class.*

The converse of Lemma 40 is not true.

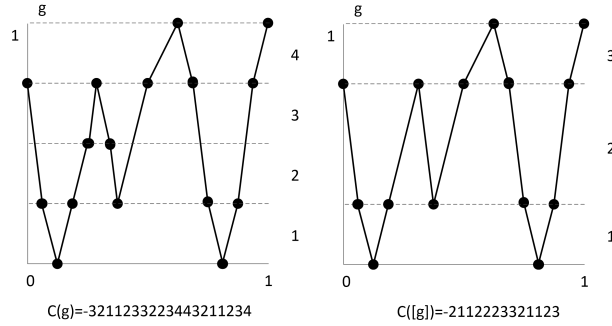


Figure 23: Example of $C(g)$ and $C([g])$.

Define the size of an evolution sequence $\pm l_1 l_2 \cdots l_n$ as $|\pm l_1 l_2 \cdots l_n| = mn$, where m represents the number of intervals $\{\mathcal{Y}_l\}$ and n the number of intervals $\{\mathcal{I}_i\}$. It is easy to show that if partition $\{\mathcal{Y}'_l\}$ is a strict refinement of partition $\{\mathcal{Y}_l\}$, then $m' > m$ and $n' > n$. Therefore, for given g , one can minimize $|\pm l_1 l_2 \cdots l_n|$ by using only the partition $\{\mathcal{Y}_l\}$ where at least one breakpoint exists at the boundary of any two adjacent intervals. In this case, the evolution sequence is referred to as the *characteristic sequence of g* , denoted by $C(g)$, as shown in Figure 23. Characteristic sequence $C(g)$ is unique for any g given the breakpoints of g .

Furthermore, not all elements in $[g]$ have the same size of the characteristic sequence, because $f_1(g(f_2))$ adds or removes type I breakpoints as seen in Section 7. The one of the minimum size is referred to as the *characteristic sequence of $[g]$* , denoted by $C([g])$. That is, suppose that

$$\hat{g}^* = \arg \min_{\hat{g} \in [g]} |C(\hat{g})|. \tag{24}$$

Then

$$C([g]) = C(\hat{g}^*). \tag{25}$$

In (24), $|C(\hat{g})|$ represents the size of $C(\hat{g})$. It may appear that if \hat{g}_1^* and \hat{g}_2^* both satisfy (24), $C(\hat{g}_1^*)$ and $C(\hat{g}_2^*)$ are not necessarily equal, even though $|C(\hat{g}_1^*)| = |C(\hat{g}_2^*)|$. If so, then $C([g])$ in (25) would not be well defined. However, Theorem 41 shows that $C(\hat{g}_1^*) = C(\hat{g}_2^*)$. Therefore, $C([g])$ defined in (25) is unique given g .

Theorem 41. *If g_1^* and g_2^* both satisfy (24), then $C(\hat{g}_1^*) = C(\hat{g}_2^*)$.*

Proof. Let $\mathcal{A} = \{0 = y_0 < y_1 < \cdots < y_m = 1\}$ be a set where if x is a type II breakpoint of g or $x = 0, 1$ then $g(x) \in \mathcal{A}$. Suppose that x_1 is a type I breakpoint and $g(x_1) \notin \mathcal{A}$. Let \mathcal{Y} be an interval such that $g(x_1) \in \mathcal{Y}$ and $\mathcal{Y} \cap \mathcal{A} = \emptyset$. Breakpoint x_1 can be eliminated by applying Corollary 20 on \mathcal{Y} . Applying Corollary 20 to eliminate all such type I breakpoints by some $f \in \mathbb{F}$, one obtains $\hat{g} = g \circ f$. When \hat{g} cannot be simplified with Corollary 20, for any interval \mathcal{Y} , either $\mathcal{Y} \cap \mathcal{A} \neq \emptyset$ or $\hat{g}^{-1}(\mathcal{Y})$ contains no breakpoints.

For \hat{g} , let partition $\{\mathcal{Y}_l\}$ where $\mathcal{Y}_l = [y_{l-1}, y_l]$ for $l = 1, \dots, m$. Characteristic sequence $C(\hat{g})$ can be obtained with partition $\{\mathcal{Y}_l\}$, because if x is a breakpoint of \hat{g} then $\hat{g}(x) \in \mathcal{A}$. On the other hand, recall that $\hat{g}_1 = f_1(g(f_2))$ with any $f_1, f_2 \in \mathbb{F}$ scales g horizontally and vertically, and thus does not add or eliminate any type II breakpoint or endpoint. The size of any partition of $[0, 1]$ to obtain $C(\hat{g}_1)$ thus cannot be smaller than $|\{\mathcal{Y}_l\}|$. Therefore, $|C(\hat{g})| \leq |C(\hat{g}_1)|$ and \hat{g} satisfies (24).

Given g , $C(\hat{g}^*)$ is unique in the sense that for $\hat{g}_1^*, \hat{g}_2^* \in [g]$, if neither \hat{g}_1^* nor \hat{g}_2^* can be simplified with Corollary 20, then $C(\hat{g}_1^*) = C(\hat{g}_2^*)$. The reason is that the partition set $\{\mathcal{Y}_l\}$ defined for \hat{g}_1^* can be scaled by some $f \in \mathbb{F}$ to become that defined for \hat{g}_2^* . As a result, \hat{g}_1^* and \hat{g}_2^* have the same evolution sequence. \square

The following corollary from Lemma 40 and Theorem 41 provides a simple way to check whether g_1 and g_2 are in the same equivalence class.

Corollary 42. $g_1, g_2 \in \mathbb{G}$ are in the same equivalence class if and only if $C([g_1]) = C([g_2])$.

9 Topological Conjugacy

Definition 17 (*Topological Conjugacy*). Continuous maps s_1 and s_2 from $[0, 1]$ to $[0, 1]$ are topologically conjugate if there exists a homeomorphism h such that $s_2 = h \circ s_1 \circ h^{-1}$.

Remark. Homeomorphism h represents a change of coordinates between s_1 and s_2 . From $s_2 = h \circ s_1 \circ h^{-1}$, it follows that $s_2^n = h \circ s_1^n \circ h^{-1}$ for $n \geq 0$. Therefore, topologically conjugate s_1 and s_2 share the same dynamics from the topological viewpoint.

It is not always an easy task to determine the topological conjugacy of s_1 and s_2 directly by the definition. However, for linear or expanding Markov maps, this task is reduced to comparing the index maps s_1^* and s_2^* associated with s_1 and s_2 . A continuous map s is *linear Markov* if it is piecewise affine and the set P of all $s^k(x)$, where $k \geq 0$ and x is an endpoint of an affine piece, is finite. A continuous map s is *expanding Markov* if it is piecewise monotone, the set P of all $s^k(x)$, where $k \geq 0$ and x is an endpoint of a monotone piece, is finite, and there is a constant $c > 1$ such that $|s(x) - s(y)| \geq c|x - y|$ whenever x and y lie in the same monotone piece. Set P is the orbit of all endpoints. The orbit of point x is defined as the set $\{s^k(x) | k \geq 0\}$ of any map s .

Let $P = \{0 = x_0 < x_1 < \dots < x_N = 1\}$ and $P^* = \{0, 1, \dots, N\}$. Define index map $s^* : P^* \rightarrow P^*$ by

$$s^*(i) = j, \text{ if } s(x_i) = x_j \tag{26}$$

for $i = 0, 1, \dots, N$.

Theorem 43 (*Block and Coven, 1987, [13, Theorem. 2.7]*). *Linear or expanding Markov maps s_1 and s_2 are topologically conjugate if and only if $s_1^* = s_2^*$ or $s_1^* = {}^*s_2$.*

Here *s is the reverse of s^* , defined by ${}^*s(i) = N - s^*(N - i)$. From Theorem 43, a linear or expanding Markov map s is characterized by s^* as far as topological conjugacy is concerned. The following proposition connects the notions of topological conjugacy and equivalence classes.

Proposition 44. *Suppose that $g_1, g_2 \in \mathbb{G}$. If $g_1^* = g_2^*$, then g_1 and g_2 are in the same equivalence class.*

Proof. Because $g_1^* = g_2^*$, $|g_1^*| = |g_2^*|$. Let $N = |g_1^*| = |g_2^*|$. For $k = 1, 2$, let $P_k = \{0 = x_{k,0} < \dots < x_{k,N} = 1\}$ be the set P of g_k . The graph of g_k is monotone on any interval $[x_{k,i-1}, x_{k,i}]$ for $i = 1, \dots, N$. Use P_1 and P_2 to derive the partition set $\{\mathcal{Y}_i\}$ defined in Section 8 to determine the evolution sequences of g_1 and g_2 . Maps g_1 and g_2 have the same evolution sequence because $g_1^* = g_2^*$, and are thus in the same equivalence class by Lemma 40. \square

The converse of Proposition 44 is not necessarily true. Figure 24 shows an example of g_1 and g_2 that are in the same equivalence class but are not topologically conjugate. In this sense, topological conjugacy is a stronger relationship between maps than equivalence classes.

Now consider topological conjugacy and λ -preservation together. Observe the following.

- Suppose that s_1 and s_2 are topologically conjugate. If $s_1 \in PA(\lambda)$, it is possible that $s_2 \notin PA(\lambda)$. An example is shown in Figure 25(a).
- It is possible that no λ -preserving s_2 exists to be topologically conjugate to a given s_1 . Figure 25(b) shows one example of such s_1 . In this example, for any topologically conjugate s_2 , $s_2(s_2(1)) = 1$, $s_2^{-1}(s_2(1)) = \{c, 1\}$ where $0 < c < s_2(1)$, and $s_2(x) < s_2(1)$ for $0 \leq x < c$ and $s_2(x) > s_2(1)$ for $c < x < 1$. Therefore, $\lambda(s_2^{-1}([s_2(1), 1])) = \lambda([c, 1]) > \lambda([s_2(1), 1])$. $s_2 \notin C(\lambda)$.

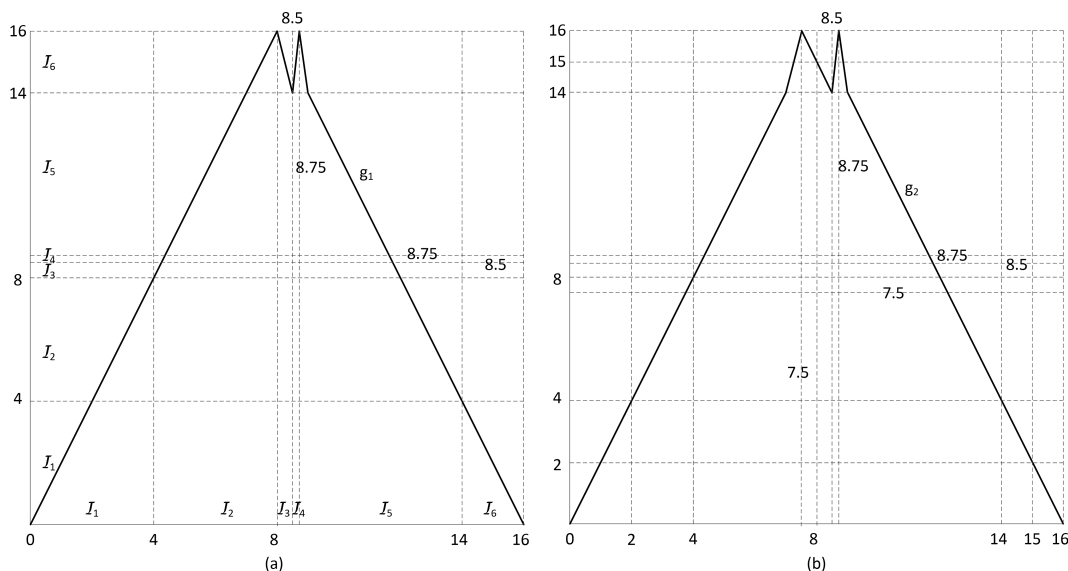


Figure 24: An example of g_1, g_2 in the same equivalence class but of distinct partition sets of P . In (a), for g_1 , $P = 2^{-4} \cdot \{0, 4, 8, 8.5, 8.75, 14, 16\}$. In (b), for g_2 , $P = 2^{-4} \cdot \{0, 2, 4, 7.5, 8, 8.5, 8.75, 14, 15, 16\}$. $g_1^* \neq g_2^*$.

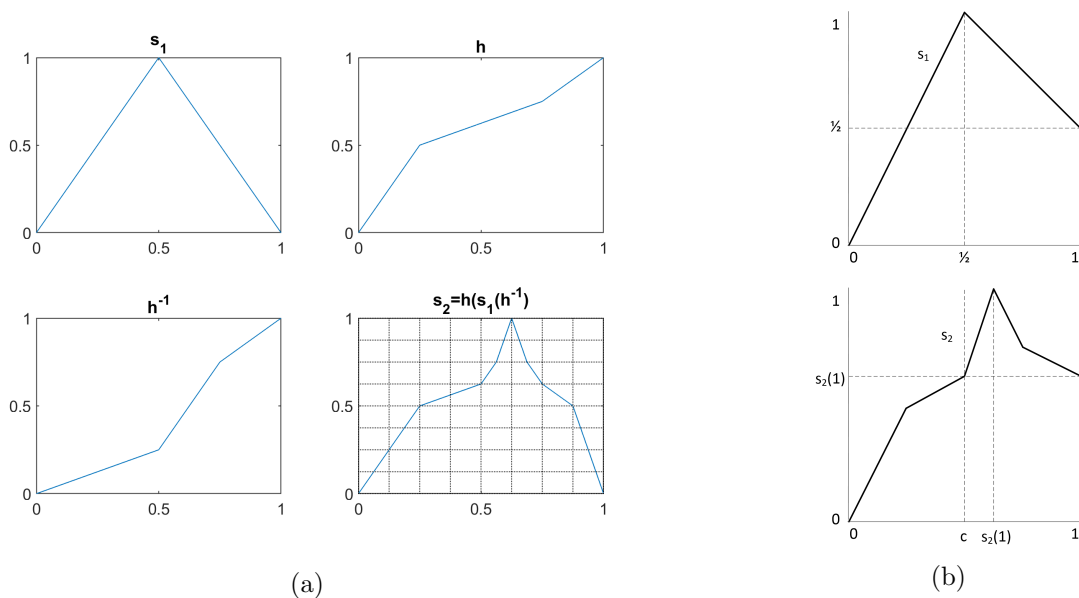


Figure 25: Two examples to illustrate the observations of topological conjugacy and λ -preservation. (a) An example shows that while $s_1 \in PA(\lambda)$, $s_2 \notin PA(\lambda)$. (b) An example of s_1 to which no λ -preserving s_2 exists that is topologically conjugate. The same homeomorphism h as in (a) is used to produce s_2 in (b) as an illustrative example.

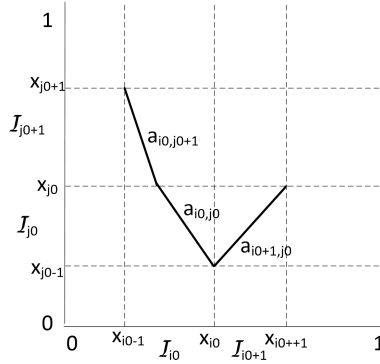


Figure 26: Illustration of piecewise expanding monotone t on \mathcal{I}_i . In this example, $s_{i_0-1}^* = j_0 + 1, s_{i_0}^* = j_0 - 1, s_{i_0+1}^* = j_0$. $A_{i_0,j_0}^* = A_{i_0,j_0+1}^* = 1$ and $A_{i_0,j}^* = 0$ for $j \neq j_0, j_0 + 1$. The graph of t is monotone on \mathcal{I}_{i_0} consisting of two affine segments with slopes a_{i_0,j_0} and a_{i_0,j_0+1} , respectively. $A_{i_0+1,j_0}^* = 1$ and $A_{i_0+1,j}^* = 0$ for $j \neq j_0$. The graph of t is affine on \mathcal{I}_{i_0+1} with slope a_{i_0+1,j_0} .

This section characterizes continuous maps s to which a λ -preserving t exists to be topologically conjugate. To this end, make the following assumption of s .

Assumption 1. First, set $\{0 = \hat{x}_0 < \dots < \hat{x}_N = 1\}$ exists such that for any $i = 0, \dots, N, s(\hat{x}_i) = \hat{x}_j$ for some j with $0 \leq j \leq N$. Second, the graph of s is monotone on $[\hat{x}_{i-1}, \hat{x}_i]$ for any $i = 1, \dots, N$.

Map s is not necessarily affine or expanding on $[\hat{x}_{i-1}, \hat{x}_i]$. Linear or expanding Markov maps are a strict subset of continuous maps that satisfy Assumption 1. The remainder of this section is to characterize continuous maps s under Assumption 1 for which $t \in PA(\lambda)$ or $t \in \mathbb{G}$ exists such that t and s are topologically conjugate. First consider linear or expanding Markov maps and then extend the results to a mixed case where the maps are linear but not expanding on some intervals and expanding but not linear on other intervals. This mixed case is important for the study of λ -preserving maps.

For s under Assumption 1, define index map s^* as in (26). The basic idea is to construct t by “continuously connecting the dots”. Specifically, let $P = \{0 = x_0 < \dots < x_N = 1\}$ be a partition of $[0, 1]$ such that $t(x_i) = x_j$ and the graph of t be a monotone and piecewise affine segment on $[x_{i-1}, x_i]$. Index map t^* is well defined in (26). The partition P and piecewise affine segments on $\{[x_{i-1}, x_i]\}$ are to be constructed so that $t^* = s^*$ and t preserves λ .

Define an $N \times N$ matrix A^* from s^*

$$A_{i,j}^* = \begin{cases} 1, & \text{if } \min(s^*(i-1), s^*(i)) < j \leq \max(s^*(i-1), s^*(i)), \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

While A^* is defined by index map s^* , s^* is uniquely determined by A^* .

Now construct piecewise affine t . Let interval $\mathcal{I}_i = [x_{i-1}, x_i]$ for $i = 1, \dots, N$. Let t be monotone on each \mathcal{I}_i and be affine on $\mathcal{I}_i \cap t^{-1}(\mathcal{I}_j)$ whenever $A_{i,j}^* = 1$. Denote by $a_{i,j}$ the slope of the affine segment. $a_{i,j} \neq 0$ is required for λ -preservation. Because t is monotone on $\mathcal{I}_i, a_{i,j} > 0$ if $s_{i-1}^* < s_i^*$ and $a_{i,j} < 0$ otherwise. See Figure 26 for an illustration. Given i , if $a_{i,j}$ is the same for all j whenever $A_{i,j}^* = 1$, then t is an affine segment on \mathcal{I}_i ; if $|a_{i,j}| > 1$ for all j whenever $A_{i,j}^* = 1$, then t is an expanding monotone piece on \mathcal{I}_i .

For $t^* = s^*, \{\mathcal{I}_i\}$ and $\{a_{i,j}\}$ are determined such that t satisfies

$$t(\mathcal{I}_i) = \bigcup_{j=\min(s^*(i-1), s^*(i))+1}^{\max(s^*(i-1), s^*(i))} \mathcal{I}_j. \tag{28}$$

That is to say that $\mathcal{I}_j \subseteq t(\mathcal{I}_i)$ if and only if $A_{i,j}^* = 1$.

Define non-negative matrix A

$$A_{i,j} = \begin{cases} |a_{i,j}|^{-1}, & \text{if } A_{i,j}^* = 1 \\ 0, & \text{otherwise.} \end{cases} \quad (29)$$

For t to be Markov and continuous and satisfy (28), $\{|\mathcal{I}_i|\}$ solves the following system of linear equations,

$$\begin{bmatrix} |\mathcal{I}_1| \\ \vdots \\ |\mathcal{I}_N| \end{bmatrix} = A \begin{bmatrix} |\mathcal{I}_1| \\ \vdots \\ |\mathcal{I}_N| \end{bmatrix}. \quad (30)$$

A vector is said *positive* if each element is positive. A solution to (30) must be a positive vector that sums to 1 for P to be a valid partition,

$$\sum_{i=1}^N |\mathcal{I}_i| = 1, |\mathcal{I}_i| > 0, \text{ for } i = 1, \dots, N. \quad (31)$$

From Lemma 2, for t to preserve λ ,

$$\sum_{i=1}^N A_{i,j} = 1 \quad (32)$$

for any j . If $\{|\mathcal{I}_1|, \dots, |\mathcal{I}_N|\}$ and $\{|a_{i,j}|\}$ exist to satisfy (30), (31) and (32), then by construction, $t \in PA(\lambda)$ and $t^* = s^*$. Furthermore, for $t \in \mathbb{G}$, $\{|a_{i,j}|\}$ are in the form of $\pm 2^k$ with integer k and $\{\mathcal{I}_i\}$ are dyadic numbers.

One can permute A by reversely mapping indices $\{1, 2, \dots, N\} \rightarrow \{N, N - 1, \dots, 1\}$ in (30). The resultant t is such that $t^* = {}^*s$. For the sake of simplicity, ignore this case of permuted A^* and *s in the remainder of this section, because such permutation does not affect the existence of t , as will be clear in Lemma 45.

From (32), A is a *column stochastic matrix* as each entry is non-negative and each column sums to 1. Thus, 1 is an eigenvalue of A . Matrix A defines a Markov chain where $A_{i,j}$ represents the transition probability from state j to i . The theory of Markov chains can be used to solve (30), (31) and (32).

Specifically, let $|\mathcal{I}_1|, \dots, |\mathcal{I}_N|$ represent the nodes of a directed graph and A^* be the adjacency matrix. An arc exists from node $|\mathcal{I}_j|$ to $|\mathcal{I}_i|$ if $A_{i,j}^* = 1$. Node $|\mathcal{I}_i|$ is *reachable* from $|\mathcal{I}_j|$ if $(A^*)_{i,j}^k > 0$ for some $k \geq 1$. Nodes $|\mathcal{I}_i|$ and $|\mathcal{I}_j|$ are said to *communicate* if $|\mathcal{I}_i|$ is reachable from $|\mathcal{I}_j|$ and $|\mathcal{I}_j|$ is reachable from $|\mathcal{I}_i|$. The set of nodes $\{|\mathcal{I}_1|, \dots, |\mathcal{I}_N|\}$ can be uniquely decomposed into K disjoint subsets C_k , with

$$\{|\mathcal{I}_1|, \dots, |\mathcal{I}_N|\} = \bigcup_{k=1}^K C_k,$$

for some positive integer K , such that nodes of each subset communicate and nodes of different subsets do not communicate. A node is *recurrent* if the probability of ever returning to node $|\mathcal{I}_i|$ starting in node $|\mathcal{I}_i|$ is 1, and is *transient* otherwise. All nodes in a given subset C_k are either recurrent or transient.

Lemma 45 (Sericola, 2013, [14]). *Let $\{\pi_1, \dots, \pi_N\}$ be the limiting probability distribution of the Markov chain defined by A . Vector $[\pi_1, \dots, \pi_N]$ is an eigenvector of A corresponding to eigenvalue 1 and sums to 1. The solution $\{\pi_1, \dots, \pi_N\}$ can be categorized into three types depending on A^* .*

- If $K = 1$, then every node of $\{|\mathcal{I}_1|, \dots, |\mathcal{I}_N|\}$ is reachable from every other node. All nodes are recurrent and A^* is said irreducible. By Perron-Frobenius Theorem [15, Theorem. 0.1], a unique positive solution $\{\pi_1, \dots, \pi_N\}$ exists.

- If $K > 1$ and every subset C_k is recurrent, then A^* can be decomposed into K irreducible subsets. Infinitely many positive solutions $\{\pi_1, \dots, \pi_N\}$ exist.
- If $K > 1$ and at least one subset C_k is transient, then no positive solution exists, because the limiting probability of any transient state is 0, therefore violating (31).

We will use Lemma 45 to study the existence of $t \in PA(\lambda)$ that is topologically conjugate to s .

First consider the case where for any j ,

$$\sum_{i=1}^N A_{i,j}^* > 1. \tag{33}$$

An example is illustrated in Figure 24(a) and described in Example 4.

Theorem 46. *If A^* is irreducible and satisfies (33), then for any $\{a_{i,j}\}$ satisfying (32), a unique t exists such that t is expanding Markov, $t \in PA(\lambda)$, and t and s are topologically conjugate if s is a linear or expanding Markov map. If A^* can be decomposed into multiple irreducible subsets, then infinitely many such t exist. If A^* can be decomposed into multiple subsets, at least one of which are transient, then no such t exists.*

Proof. From (33), given any j , the number of i for which $A_{i,j}^* = 1$ for $i = 1, \dots, N$ is greater than 1. Thus it is easy to select $\{|a_{i,j}|\}$, with $|a_{i,j}| > 1$ for any i, j , to satisfy (32). The graph of t on \mathcal{I}_i is thus expanding for any i .

Consider the first case of A^* being irreducible. From Lemma 45, a unique positive eigenvector \mathbf{v} of A exists corresponding to eigenvalue 1 with $|\mathbf{v}| = 1$. Let $[|\mathcal{I}_1|, \dots, |\mathcal{I}_N|]^T = \mathbf{v}$. Let partition set $P = \{0 = x_0 < \dots < x_N = 1\}$ where $x_i = \sum_{l=1}^i |\mathcal{I}_l|$ for $i = 1, \dots, N$. Let $t(x_i) = x_{s_i^*}$. Interval \mathcal{I}_i is partitioned into $\{\mathcal{I}_{i,j_0} < \dots < \mathcal{I}_{i,j_1}\}$ where $A_{i,j}^* = 1$ for $j = j_0, \dots, j_1$. If $s_{i-1}^* < s_i^*$, then $|\mathcal{I}_{i,j}| = |a_{i,j}|^{-1} |\mathcal{I}_j|$ and t is an affine segment with slope $a_{i,j}$; If $s_{i-1}^* > s_i^*$, then $|\mathcal{I}_{i,j}| = |a_{i,j}|^{-1} |\mathcal{I}_{j_1+j_0-j}|$ and t is an affine segment with slope $-a_{i,j}$. Therefore t is completely defined. By (30), t is continuous at the boundary between the piecewise affine segments on adjacent intervals \mathcal{I}_{i-1} and \mathcal{I}_i . By construction, t is expanding Markov, $t^* = s^*$ and $t \in PA(\lambda)$. If s is a linear or expanding Markov map, then by Theorem 43, t and s are topologically conjugate.

The other two cases of A^* can be shown analogously. □

Additional conditions are required for $t \in \mathbb{G}$ as stated in the following corollary.

Corollary 47. *Suppose that A^* is irreducible and satisfies (33). For a set of $\{a_{i,j}\}$ satisfying (32) and being in the form of $\pm 2^k$ for integer k , if (30) and (31) have a dyadic solution, then unique t exists such that t is expanding Markov, $t \in \mathbb{G}$, and t and s are topologically conjugate.*

The following examples show that the choice of $\{a_{i,j}\}$ makes no difference in determining whether $t \in PA(\lambda)$ exists, as expected from Theorem 46, but plays an important role for $t \in \mathbb{G}$.

Example 4. Suppose that $N = 6$, $s^*(0) = 0, s^*(1) = 2, s^*(2) = 6, s^*(3) = 5, s^*(4) = 6, s^*(5) = 1, s^*(6) = 0$. From s^* ,

$$A^* = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A^* is irreducible. Let

$$A = \begin{bmatrix} 2^{-1} & 2^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} \\ 0 & 0 & 0 & 0 & 0 & 2^{-2} \\ 0 & 0 & 0 & 0 & 0 & 2^{-3} \\ 0 & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-3} \\ 2^{-1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution to (30) is given by

$$\left[|\mathcal{I}_1|, \dots, |\mathcal{I}_6| \right] = \left[\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{32} \quad \frac{1}{64} \quad \frac{21}{64} \quad \frac{1}{8} \right].$$

In this case, $t \in \mathbb{G}$. Figure 24(a) plots t . However, for a slightly different choice of A

$$A = \begin{bmatrix} 2^{-1} & 2^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-2} \\ 0 & 0 & 0 & 0 & 0 & 2^{-1} \\ 0 & 0 & 0 & 0 & 0 & 2^{-3} \\ 0 & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-1} & 2^{-3} \\ 2^{-1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution to (30) is given by

$$\left[|\mathcal{I}_1|, \dots, |\mathcal{I}_6| \right] = \left[\frac{4}{17} \quad \frac{4}{17} \quad \frac{1}{17} \quad \frac{1}{68} \quad \frac{23}{68} \quad \frac{2}{17} \right].$$

In this case, $t \in PA(\lambda)$ but $t \notin \mathbb{G}$.

Example 5. A^* is not irreducible in the following two cases. First,

$$A^* = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

No positive solution to (30) exists, because solving (30) for any $\{a_{i,j}\}$ leads to $|\mathcal{I}_1| = |\mathcal{I}_2| = |\mathcal{I}_3| = 0$ because the first three states are transient. Next,

$$A^* = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Infinitely many positive solutions to (30) exist for any $\{a_{i,j}\}$, because the first two states form an irreducible subset and the last two states form another irreducible subset.

Next consider the case where for some j_0 ,

$$\sum_{i=1}^N A_{i,j_0}^* = 1. \tag{34}$$

The difference from (33) is that in (34) a single i_0 exists such that $A_{i_0,j_0}^* = 1$ and $A_{i,j_0}^* = 0$ for all $i \neq i_0$. From (32), $|a_{i_0,j_0}| = 1$ to be λ -preserving. If $A_{i_0,j'}^* = 1$ for some $j' \neq j_0$, then t on \mathcal{I}_{i_0} is neither an affine segment nor an expanding monotone piece as illustrated in Figure 27(a), in which case Theorem 43 cannot be applied. To avoid this problem, the remainder of the section assumes the following.

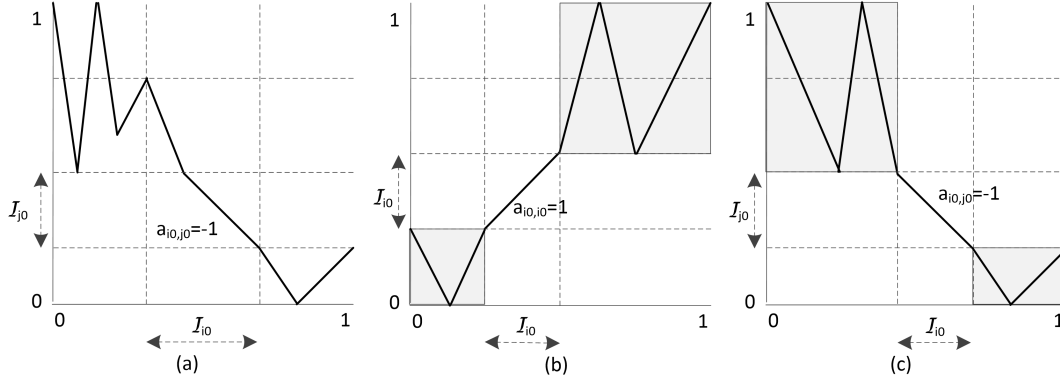


Figure 27: Illustration of the cases described by (34). Case (a) is not considered under Assumption 2. Cases (b) and (c) are addressed in Corollary 48 and Corollary 50 respectively.

Assumption 2. First, one and only one j_0 exists such that a single i_0 exists where $A_{i_0, j_0}^* = 1$ and $A_{i, j_0}^* = 0$ for all $i \neq i_0$. Second, for i_0 obtained in part (1), $A_{i_0, j}^* = 0$ for any $j \neq j_0$.

Part (1) of Assumption 2 is for simplicity. The results can be easily extended to the case where multiple such j_0 exist. Part (2) is needed such that t on any \mathcal{I}_i is either expanding monotone or an affine segment with slope equal to ± 1 . Specifically, a unique pair of indices i_0, j_0 exist such that

$$\begin{cases} |a_{i_0, j_0}| = 1, \\ |a_{i, j}| > 1, \quad \text{whenever } A_{i, j}^* = 1 \text{ and if } i \neq i_0 \text{ or } j \neq j_0. \end{cases} \quad (35)$$

Even with Assumption 2, Theorem 43 cannot be directly applied because t is not a linear or expanding Markov map: t is not expanding on \mathcal{I}_{i_0} and not necessarily¹ linear on \mathcal{I}_i for $i \neq i_0$. Next, we apply Definition 17 of topological conjugacy directly to circumvent this technical issue.

First, suppose that $s_{i_0-1}^* < s_{i_0}^*$, as illustrated in Figure 27(b). Then $a_{i_0, j_0} = 1$. In this case,

$$t \left(\bigcup_{i=1}^{i_0-1} \mathcal{I}_i \right) = \bigcup_{i=1}^{j_0-1} \mathcal{I}_i \text{ and } t^{-1} \left(\bigcup_{i=1}^{j_0-1} \mathcal{I}_i \right) = \bigcup_{i=1}^{i_0-1} \mathcal{I}_i.$$

j_0 must be equal to i_0 , because

$$\lambda \left(\bigcup_{i=1}^{j_0-1} \mathcal{I}_i \right) = \sum_{i=1}^{j_0-1} |\mathcal{I}_i| = \lambda \left(\bigcup_{i=1}^{i_0-1} \mathcal{I}_i \right) = \sum_{i=1}^{i_0-1} |\mathcal{I}_i|.$$

Therefore,

$$\begin{cases} s_i^* \leq i_0 - 1, & \text{if } i < i_0 - 1, \\ s_i^* = i, & \text{if } i = i_0 - 1, i_0 \\ s_i^* \geq i_0, & \text{if } i > i_0. \end{cases} \quad (36)$$

Figure 28 shows the construction of t . Recall that $\{0 = \hat{x}_0 < \dots < \hat{x}_N = 1\}$ is the partition of s . First, revise s to obtain s' by eliminating the portion on $[\hat{x}_{i_0-1}, \hat{x}_{i_0}]$, scaling up the portion on $[0, \hat{x}_{i_0-1}]$ by a factor of $\frac{1}{\alpha}$ to fill up $[0, \hat{x}_{i_0}]$ and keeping the portion on $[\hat{x}_{i_0}, 1]$ unchanged. Specifically,

$$s'(x) = \begin{cases} \frac{1}{\alpha} s(\alpha x), & \text{if } 0 \leq x < \hat{x}_{i_0}, \\ s(x), & \text{otherwise.} \end{cases} \quad (37)$$

¹For t to be a λ -preserving linear Markov map, $A_{i, j}$ must be the same for all j given i whenever $A_{i, j}^* = 1$ and (32) must be satisfied. Whether $\{a_{i, j}\}$ exists to meet both conditions depends on A^* . For the first two A^* in Example 6 t is a linear Markov map, and for the last A^* t is not a linear or expanding Markov map.

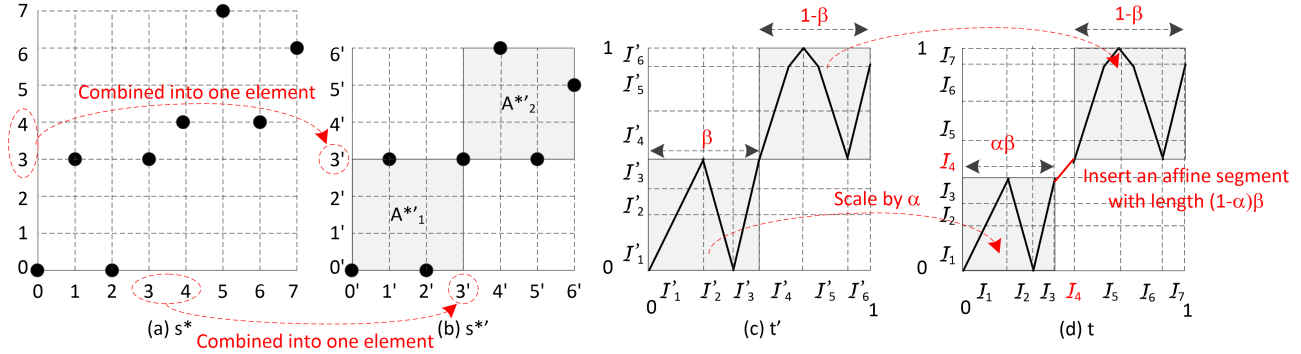


Figure 28: Construction of t in the case of (34) with $a_{i_0, j_0} = 1$. Here $N = 7$ and $i_0 = 4$.

with

$$\alpha = \frac{\hat{x}_{i_0-1}}{\hat{x}_{i_0}}. \quad (38)$$

From (36), $s(\hat{x}_{i_0-1}) = \hat{x}_{i_0-1}$ and $s(\hat{x}_{i_0}) = \hat{x}_{i_0}$. In (37), $s'((\hat{x}_{i_0})^-) = \frac{1}{\alpha}s(\hat{x}_{i_0-1}) = s(\hat{x}_{i_0}) = s'((\hat{x}_{i_0})^+)$. Thus, s' is continuous and expanding Markov. The change from s to s' in effect combines elements $i_0 - 1$ and i_0 of the $(N + 1)$ -element set $P^* = \{0, 1, \dots, N\}$ into a single element to arrive at an N -element set $P^{*'} = \{0', 1', \dots, (N - 1)'\}$ where 0 becomes $0'$, 1 becomes $1'$, and so on, and $i_0 - 1$ and i_0 are combined to become $(i_0 - 1)'$, then $i_0 + 1$ becomes i'_0 , $i_0 + 2$ becomes $(i_0 + 1)'$, and so on, and N becomes $(N - 1)'$. Revise the index map s^* to become $s^{*'}$ when P^* becomes $P^{*'}$, as shown in Figure 28(a) and (b).

The revised $A^{*'}$ obtained from the revised $s^{*'}$ satisfies (33) instead of (34). Matrix $A^{*'}$ is a block-diagonal one consisting of $A^{*'}_1$ and $A^{*'}_2$ where $A^{*'}_1$ is a map of $\{\mathcal{I}'_1, \dots, \mathcal{I}'_{i_0-1}\}$ to itself and $A^{*'}_2$ is a map of $\{\mathcal{I}'_{i_0}, \dots, \mathcal{I}'_{N-1}\}$ to itself, as shown in Figure 28(c). If $A^{*'}$ can be decomposed into multiple irreducible subsets, then from Theorem 46, infinitely many t' exists such that t' is expanding Markov, $t' \in PA(\lambda)$, and t' and s' are topologically conjugate. Set $\{\mathcal{I}'_1, \dots, \mathcal{I}'_{N-1}\}$ is obtained in the construction of t' according to Theorem 46. Let homeomorphism h' be such that

$$s' = h' \circ t' \circ h'^{-1}. \quad (39)$$

Comparing the scaling (37) with (39), it follows that

$$[\hat{x}_{i-1}, \hat{x}_i] = \begin{cases} \alpha h'(\mathcal{I}'_i), & \text{for } i = 1, \dots, i_0 - 1, \\ h'(\mathcal{I}'_{i-1}), & \text{for } i = i_0 + 1, \dots, N. \end{cases}$$

In particular, $\hat{x}_{i_0} = h'(\beta)$ where $\beta = \sum_{i=1}^{i_0-1} |\mathcal{I}'_i|$.

Finally, revise t' to obtain t by scaling down $\mathcal{I}'_1, \dots, \mathcal{I}'_{i_0-1}$ by a factor of α , keeping $\mathcal{I}'_{i_0}, \dots, \mathcal{I}'_{N-1}$ unchanged, and inserting a new interval of length $(1 - \alpha)\beta$ between \mathcal{I}'_{i_0-1} and \mathcal{I}'_{i_0} to arrive at a set of N intervals $\mathcal{I}_1, \dots, \mathcal{I}_N$ with $\sum_{i=1}^N |\mathcal{I}_i| = 1$, as shown in Figure 28(d). Let t be an affine segment with slope 1 on \mathcal{I}_{i_0} and continuous between $\mathcal{I}_{i_0-1}, \mathcal{I}_{i_0}$ and between $\mathcal{I}_{i_0}, \mathcal{I}_{i_0+1}$. Revise h' to obtain h correspondingly

$$h(x) = \begin{cases} \alpha h' \left(\frac{1}{\alpha} x \right), & \text{if } 0 \leq x < \alpha\beta, \\ \hat{x}_{i_0} + \frac{\hat{x}_{i_0}}{\beta} (x - \beta), & \text{if } \alpha\beta \leq x < \beta, \\ h'(x), & \text{otherwise.} \end{cases} \quad (40)$$

Recall that t on \mathcal{I}_{i_0} is affine with slope 1. If s is affine with slope 1 on $[\hat{x}_{i_0-1}, \hat{x}_{i_0}]$, then by (39) and (40), it follows that

$$s = h \circ t \circ h^{-1}. \quad (41)$$

Then

$$[\hat{x}_{i-1}, \hat{x}_i] = h(\mathcal{I}_i)$$

for $i = 1, \dots, N$. Moreover, because $t' \in PA(\lambda)$ and t on \mathcal{I}_{i_0} is affine with slope 1, $t \in PA(\lambda)$. Hence, the following corollary holds.

Corollary 48. *Suppose that A^* satisfies (34) and that Assumption 2 holds. Suppose that $j_0 = i_0$ and $s_{i_0-1}^* < s_{i_0}^*$. Suppose that s is affine with slope 1 on $[\hat{x}_{i_0-1}, \hat{x}_{i_0}]$. If the revised A^* can be decomposed into irreducible subsets, then t exists such that t and s are topologically conjugate and $t \in PA(\lambda)$. Furthermore, suppose that t' is constructed from the revised s' in (37) and (39). If $t' \in \mathbb{G}$ and α of (38) is dyadic, then $t \in \mathbb{G}$.*

Clearly t is not unique in the preceding construction because the solution of $\{\mathcal{I}'_1, \dots, \mathcal{I}'_{N-1}\}$ in Theorem 46 are not unique.

Next, suppose that $s_{i_0-1}^* > s_{i_0}^*$, as illustrated in Figure 27(c). Then $a_{i_0, j_0} = -1$. If $j_0 = i_0$, then one can construct t analogously to the preceding case of $a_{i_0, j_0} = 1$ except that in the final step when a new interval of length $(1 - \alpha)\beta$ is inserted as shown in Figure 28(d), t is an affine segment with slope -1 , instead of 1, on \mathcal{I}_{i_0} .

Now suppose $j_0 \neq i_0$. Assume $i_0 < j_0$. (The case of $i_0 > j_0$ can be addressed analogously.) In this case,

$$t(\mathcal{I}_{i_0}) = \mathcal{I}_{j_0} \text{ and } t^{-1}(\mathcal{I}_{j_0}) = \mathcal{I}_{i_0}. \tag{42}$$

$$t \left(\bigcup_{i=1}^{i_0-1} \mathcal{I}_i \right) = \bigcup_{i=j_0+1}^N \mathcal{I}_i \text{ and } t^{-1} \left(\bigcup_{i=j_0+1}^N \mathcal{I}_i \right) = \bigcup_{i=1}^{i_0-1} \mathcal{I}_i. \tag{43}$$

λ -preservation leads to

$$|\mathcal{I}_{j_0}| = |\mathcal{I}_{i_0}|. \tag{44}$$

$$\lambda \left(\bigcup_{i=j_0+1}^N \mathcal{I}_i \right) = \sum_{i=j_0+1}^N |\mathcal{I}_i| = \lambda \left(\bigcup_{i=1}^{i_0-1} \mathcal{I}_i \right) = \sum_{i=1}^{i_0-1} |\mathcal{I}_i|. \tag{45}$$

Lemma 49. *Suppose that A^* satisfies (34) and that Assumption 2 holds. Suppose that $a_{i_0, j_0} = -1$ and $i_0 < j_0$. Let $\{0 = x_0 < \dots < x_N = 1\}$ be a partition of $[0, 1]$ and $\mathcal{I}_i = [x_{i-1}, x_i]$ for $i = 1, \dots, N$. Let $t \in C(\lambda)$ be monotone on each \mathcal{I}_i and be affine on $\mathcal{I}_i \cap t^{-1}(\mathcal{I}_j)$ where $A_{i,j}^* = 1$, and the slope $a_{i,j}$ of the affine segment satisfies (35). Then*

$$\left| \frac{d(t \circ t)}{dx} \right| > 1 \tag{46}$$

for all $x \in [0, 1]$ except for a finite number of points.

Proof. From (44) and (45), it is easy to show that intervals \mathcal{I}_{i_0} and \mathcal{I}_{j_0} are symmetric with respect to $\frac{1}{2}$. From (35), $\left| \frac{dt}{dx} \right| = 1$ on \mathcal{I}_{i_0} and $\left| \frac{dt}{dx} \right| > 1$ on $[0, 1] \setminus \mathcal{I}_{i_0}$.

The affine segment of t with slope -1 on \mathcal{I}_{i_0} affects $t \circ t$ in two ways, as illustrated in Figure 29. First, t on \mathcal{I}_{j_0} flips horizontally along the $x = \frac{1}{2}$ axis to become $t \circ t$ on \mathcal{I}_{i_0} . Second, t on $t^{-1}(\mathcal{I}_{i_0})$ flips vertically along the $y = \frac{1}{2}$ axis to become $t \circ t$ on $(t \circ t)^{-1}(\mathcal{I}_{j_0})$. On other portions of $[0, 1]$, $t \circ t$ is obtained by the composition of two segments each with $\left| \frac{dt}{dx} \right| > 1$, and thus $\left| \frac{d(t \circ t)}{dx} \right|$ is greater than either of the two. Hence, (46) holds on $[0, 1]$ whenever the derivative is defined. \square

Lemma 49 states that $t \circ t$ is expanding Markov although t is not. The proof of the expanding Markov case of Theorem 43 in [13, Theorem. 2.1] notes that the expanding property of t is used only to make

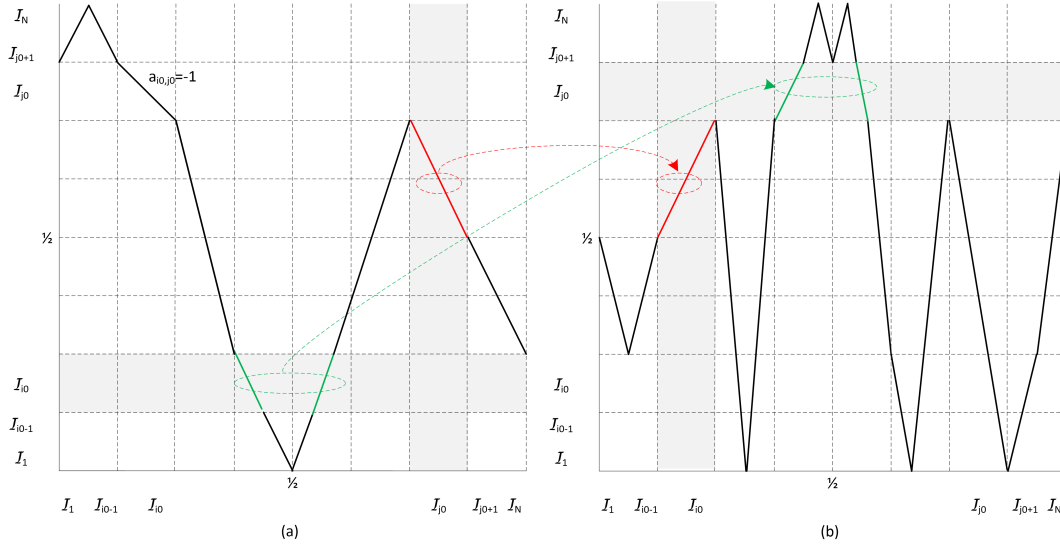


Figure 29: Illustration of t in (a) and $t \circ t$ in (b) where t satisfies (42) and (43). The graph of t on \mathcal{I}_{i_0} is an affine segment of slope -1 . $t(\mathcal{I}_{i_0}) = \mathcal{I}_{j_0}$. This affine segment transforms portions of t to become portions of $t \circ t$ as highlighted as green and red segments.

$\bigcup_n t^{-n}(P)$ dense and the theorem holds for maps for which some power is expanding Markov. Hence, the following corollary holds.

Corollary 50. *Suppose that A^* satisfies (34) and that Assumption 2 holds. Suppose that $a_{i_0, j_0} = -1$. If $i_0 = j_0$, then the conclusion of Corollary 48 holds. If $i_0 \neq j_0$, then the conclusions of Theorem 46 and Corollary 47 hold.*

Example 6. First, infinitely many $t \in PA(\lambda)$ exist in Corollaries 48 and 50, respectively, for

$$A^* = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

where $i_0 = j_0 = 3$. $t \in \mathbb{G}$ exists. Second, a unique $t \in PA(\lambda)$ exists in Corollary 50 for

$$A^* = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $i_0 = 6, j_0 = 1$. However, in this case, $t \notin \mathbb{G}$.

10 Conclusion and Future Study

This paper has introduced a new and interesting monoid, λ -preserving Thompson’s monoid \mathbb{G} , modeled on Thompson’s group \mathbb{F} , and studied a number of properties of \mathbb{G} . The main results of this paper improve several results of [3] and demonstrate an interesting interplay between algebraic and dynamical settings.

A few areas are worth further exploring.

First, ergodicity. This paper has studied the properties of TM, LEO and Markov, and characterized periods of periodic points of maps in \mathbb{G} . The next step is to systematically study statistical properties of long-term time averages of various functions along trajectories of the dynamical system governed by $g \in \mathbb{G}$.

Second, presentations. Thompson's group \mathbb{F} admits infinite and finite presentations. As seen in this paper, λ -preserving Thompson's monoid \mathbb{G} is more sophisticated and the notions of equivalence classes and sets of equivalence classes are useful in presenting \mathbb{G} . The next step is to construct various presentations of \mathbb{G} .

Third, analogue of \mathbb{G} in a high dimensional space. This paper has studied interval maps, which exist in a one-dimensional space. It will be interesting to extend the study to a higher dimensional space and see if different conclusions will be drawn for high dimensional maps as opposed to interval maps.

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