Product Expansions of q-Character Polynomials

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Abstract

We consider certain class functions defined simultaneously on the groups $\mathrm{Gl}_n(\mathbb{F}_q)$ for all n, which we also interpret as statistics on matrices. It has been previously shown that these simultaneous class functions are closed under multiplication, and we work towards computing the structure constants of this ring of functions. We derive general criteria for determining which statistics have nonzero expansion coefficients in the product of two fixed statistics. To this end, we introduce an algorithm that computes expansion coefficients in general, which we furthermore use to give closed form expansions in some cases. We conjecture that certain indecomposable statistics generate the whole ring, and indeed prove this to be the case for statistics associated with matrices consisting of up to 2 Jordan blocks. The coefficients we compute exhibit surprising stability phenomena, which in turn reflect stabilizations of joint moments as well as multiplicities in the irreducible decomposition of tensor products of representations of finite general linear groups.

1 Introduction

Statistics of random matrices and representations of finite matrix groups are often of great interest in mathematics. These topics were studied extensively by many people, and they contain applications to areas of mathematics including group theory and random number generators (see for example the work of Fulman [3, 4]).

Class functions are a central aspect in this field of study. When considering matrices in $\mathrm{Gl}_n(\mathbb{F}_q)$ for any n and q, that is, invertible matrices with entries in finite fields, one simple class function is $\mathrm{Fix}(A)$, which counts the number of nonzero vectors fixed under A. Put explicitly, $\mathrm{Fix}(A) = |\{v \in \mathbb{F}_q^n \mid Av = v \text{ and } v \neq 0\}|$, where A is an invertible $n \times n$ matrix with entries in the finite field $\mathbb{F}_q$. A natural question to consider centers on finding the distribution of $\mathrm{Fix}(A)$ as A ranges uniformly over all $\mathrm{Gl}_n(\mathbb{F}_q)$. For example, it has been shown that the expectation is 1 (see [7, Corollary 1.3]). Fulman and Stanton [5] previously determined moments of this class function through generating functions. Surprisingly, the expectation of $\mathrm{Fix}(\cdot)^k$ is independent of n once $n \geq k$.

Gadish extended this notion from fixed vectors to fixed subspaces in [6, Definition 2.5], where he defined a collection of class functions X_B with domain $\mathrm{Gl}_n(\mathbb{F}_q)$ for all n simultaneously, known as *q-character polynomials*. This extension of the number of nonzero fixed vectors preserves the property that expectations of all powers are independent of n for sufficiently large n. We first provide a definition for the class functions being considered.

Definition 1 (q-character polynomials)**.** Given a finite field $\mathbb{F}_q$, let B be an $m \times m$ matrix. Let X_B be the following function on the collection of square matrices of any dimension over $\mathbb{F}_q$: if A is any $n \times n$ matrix

$$X_B(q, A) = |\{W \leq \mathbb{F}_q^n \mid \dim W = m \text{ with } A(W) \subseteq W \text{ and } A|_W \sim B\}|.$$

Here $A|_W \sim B$ refers to matrix similarity. Because the matrices A and B are also defined over larger fields F_{q^d}, we could also consider $X_B(q^d, A)$ for any $d > 1$. Unless otherwise specified, the size of the field is assumed to be fixed at some prime power q, and we use the simplified notation $X_B(A)$ to mean $X_B(q, A)$.

It is clear from the definition that q-character polynomials are indeed class functions. Gadish [6, 7] previously showed that the functions X_B span a ring.

Theorem 1.1 ([6, 7]). *Given invertible matrices B_1 and B_2 of size k_1 and k_2 respectively, there exists an expansion for the pointwise product $X_{B_1} \cdot X_{B_2}$:*

$$X_{B_1} \cdot X_{B_2} = \sum_C \lambda^C_{B_1,B_2} X_C \tag{1}$$

for some scalars $\lambda^C_{B_1,B_2}$, where the sum ranges over conjugacy classes of invertible matrices C of size $\max(k_1, k_2) \leq k \leq k_1 + k_2$.

However, the product expansion coefficients of two fixed statistics are not known in general.

Goal. This paper seeks to characterize the scalars $\lambda^C_{B_1,B_2}$ that are associated with each product and describe their properties.

We note below (see §5.4) that some special values of the functions X_B are q-binomial coefficients, also known as Gaussian binomial coefficients. Thus, in the same way that products of binomial coefficients and q-binomial coefficients occur naturally in mathematics and are inherently interesting objects of study, our statistics yield a generalization of these products. Our results can also be used to obtain a stable formula for the factorization of the tensor products of representations, although we do not pursue this particular application in any detail here.

As a first example, we compute the following product expansion.

Example 1.1.1. *Let J_k be the unipotent Jordan block of size k and let $J_{k,\ell}$ be the block matrix consisting of blocks J_k and J_ℓ. For all $n > 3$,*

$$X_{J_n} \cdot X_{J_3} = X_{J_n} + q(q-1)X_{J_{n,1}} + q^3(q-1)X_{J_{n,2}} + q^6 X_{J_{n,3}}.$$

For more extensive calculations of product expansions, see Section 5 for general calculations for matrices with 1 or 2 Jordan blocks and Appendix A for a table of specific calculations in small cases.

Remark 1.1.1. The reader surely observes that the expansion coefficients in Example 1.1.1 are polynomials in q with integer coefficients, and are independent of n. This pattern persists in general, as we prove as part of this work.

Similar stabilization of the product expansion coefficients will be seen in more general expansions in Section 5.

Main Results

To find product expansions of functions X_B, we develop a recursive algorithm in Section 2. A key tool in applying this algorithm is our characterization of evaluations of the functions X_B. This mainly take the form of our Evaluation Formula, given in Section 4, for evaluating $X_B(A)$ for any two unipotent matrices A and B. Along side that formula we prove the following two general results.

Theorem 1.2. *For every two invertible matrices A and B over $\mathbb{F}_q$, there exists a polynomial $P_{A,B}(t)$ with integer coefficients such that $X_B(q^d, A) = P_{A,B}(q^d)$ for every positive integer d, where $X_B(q^d, A)$ counts subspaces over $\mathbb{F}_{q^d}$.*

Moreover, suppose that A and B are unipotent matrices in Jordan form, thus defined over $\mathbb{F}_p$ for every prime p. Then the same polynomial $P_{A,B}(t)$ satisfies $X_B(p^d, A) = P_{A,B}(p^d)$ for any prime power p^d. I.e. the polynomial $P_{A,B}(t)$ does not depend on the field.

Theorem 1.2 demonstrates that X_B is in fact a polynomial in q and allows us to compute the statistic X_B when changing the size of the field. Theorem 1.3 below relates the general polynomials $P_{A,B}(t)$ to polynomials for pairs of unipotent matrices. With that, we conclude that one must only consider statistics on unipotent matrices to determine general product expansions.

Theorem 1.3. *For every pair of partitions λ and μ there exists a polynomial $P_{\lambda,\mu}(t) \in \mathbb{Z}[t]$, such that for every pair of invertible matrices A and B over $\mathbb{F}_q$,*

$$P_{A,B}(t) = \prod_i P_{\lambda_i,\mu_i}(t^{r_i})$$

where the partitions λ_i and μ_i denote the respective Jordan block sizes of A and B with common eigenvalue ζ_i, for $\zeta_1, \ldots, \zeta_k$ non-Galois-conjugate representatives of the common eigenvalues of A and B with minimal polynomials of respective degrees $r_1, \ldots, r_k$.

We further use our algorithm to show the following general expansion results for unipotent matrices.

Theorem 1.4. *Let A and B be unipotent matrices. The product expansion coefficients of $X_A \cdot X_B$ as in Equation (1) are polynomials in q with integer coefficients.*

We also find that many product expansion coefficients vanish automatically, or don't depend too heavily on the largest Jordan block sizes. See the following proposition for example, and find more results in this vein in Section 5.

Proposition 1.5. *Let A and B be unipotent Jordan matrices such that the largest Jordan blocks of A and B are a and b, respectively, with $a \leq b$. Consider the expansion*

$$X_A \cdot X_B = \sum_C \lambda_{A,B}^C X_C.$$

If $\lambda_{A,B}^C \neq 0$, then the largest Jordan block in C has size b.

Lastly, our product expansion calculations serve as a tool for evaluating joint moments of the random variables X_B. E.g. one easily deduces that the numbers of eigenvectors with distinct eigenvalues λ_1 and λ_2 are uncorrelated among random matrices of size at least 2×2. One more significant application of our calculations is the determination of the correlation between the random variables X_{J_a} – counting the number of a-dimensional subspaces on which a random matrix acts unipotently and indecomposably.

Theorem 1.6. *The correlation between the random variables X_{J_a} and X_{J_b} in the uniform probability space $\mathrm{Gl}_n(\mathbb{F}_q)$ is $\sqrt{\frac{q^a-1}{q^b-1}}$ where $b \geq a$ and for all $n \geq 2b$.*

1.1 Organization

In Section 2, we outline a general procedure with which to approach the problem of finding the product expansion scalars through an inductive method. In Section 3, we simplify the problem at hand by noting a few significant lemmas which allow us to write all statistics in the form of statistics on unipotent matrices with a Jordan form. In Section 4, we determine an explicit formula for evaluating statistics at any given matrix using a basis and space-picking scheme.

Section 5 contains most of our substantial results. Namely, we prove the scalars in Theorem 1.1 are polynomials in q; we show for unipotent matrices A and B that if X_C has a nonzero coefficient in the product expansion $X_A \cdot X_B$, then the maximal Jordan block size in C is equal to the maximal Jordan block size across A and B; we employ our evaluation formula to calculate the expansion coefficients for products of statistics on matrices composed of single Jordan blocks and propose possible coefficients for products of statistics on matrices composed of multiple Jordan blocks. Lastly, using a combinatorial argument, we examine the product expansion of statistics on identity matrices.

In Section 6, we use the results from Section 5 to compute joint moments of X_B, thought of as random variables. We explicitly determine the correlation coefficient between each pair of statistics of single Jordan blocks based on our expansions in Section 5.

Lastly, in Appendix A, we provide a comprehensive list of calculated product expansions that illustrate key properties and go beyond the general formulas provided in Section 5.

2 Algorithm

In this section, we describe a procedure for determining the expansion coefficients by evaluating certain statistics at a series of specified matrices.

Notation. For a square matrix S, let $\dim(S)$ denote the number of rows of S.

We seek to determine the coefficients $\lambda^C_{B_1,B_2}$ in the expansion $X_{B_1} \cdot X_{B_2}$. One key observation in the calculation process is the following:

Proposition 2.1. *Given two matrices A, B where $\dim(A) \le \dim(B)$, $X_B(A) = 0$ unless $A \sim B$, in which case $X_B(A) = 1$.*

Proof. If we consider any subspace $W \subseteq \mathbb{F}^{\dim(A)}$, then $\dim(W) \le \dim(A) \le \dim(B)$. By the definition of X_B, $\dim(W) = \dim(B)$ so $\dim(W) = \dim(A) = \dim(B)$ or else $X_B(A) = 0$. If $X_B(A) \neq 0$, then W must be $\mathbb{F}^{\dim(A)}$, so the condition $A|_W \sim B$ simplifies to $A \sim B$. In this case, only one subspace is counted so $X_B(A) = 1$. ∎

By evaluating the statistics in the equation described in Theorem 1.1 at a given conjugacy class of a matrix C, we see that

$$X_{B_1}(C) \cdot X_{B_2}(C) = \lambda^C_{B_1,B_2} X_C(C) + \sum_{\dim(M)<\dim(C)} \lambda^M_{B_1,B_2} X_M(C).$$

In other words,

$$\lambda^C_{B_1,B_2} = X_{B_1}(C) \cdot X_{B_2}(C) - \sum_{\dim(M)<\dim(C)} \lambda^M_{B_1,B_2} X_M(C), \tag{2}$$

allowing us to use an inductive procedure to calculate the coefficients.

- First, starting from $k = \max\{\dim(B_1), \dim(B_2)\}$, we consider the conjugacy classes of k-dimensional matrices.
- Next, for each conjugacy class of matrices of dimension k, we consider the Jordan matrix C in that conjugacy class (possibly passing to a larger field, but the effect of this field extension on evaluations is known by Theorem 1.2).
- Then, we evaluate X_{B_1}, X_{B_2}, and X_M where $\dim(M) < \dim(C)$, and using Eq. (2), we can determine $\lambda^C_{B_1,B_2}$.
- Using this procedure to calculate the coefficients for matrices of a given dimension k, we increment k by 1 and repeat the previous steps until $k = \dim(B_1)+\dim(B_2)$, thus determining all of the coefficients we seek.

It follows that to find the product expansion coefficients, one approach is to find a general formula for evaluating any given statistic on a specified matrix.

3 Reductions

In this section, we examine several reductions that simplify the problem to one concerning statistics of unipotent Jordan matrices.

3.1 Relating General Matrices with those in Jordan form

We provide a method to determine all statistics X_B in terms of statistics on matrices that have a Jordan form. Consider the following simple example.

Example 3.0.1. *Recall that over the field $\mathbb{F}_2$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^2 \end{pmatrix}$ where $\epsilon \in \mathbb{F}_4$ satisfies $\epsilon^2+\epsilon+1 = 0$. Then, $X_{\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)}(2, A) = X_{(\epsilon)}(4, A)$ for every matrix A over $\mathbb{F}_2$.*

We let F denote the Frobenius automorphism. In general, we have the following result.

Theorem 3.1. *Let M_0 be a nilpotent matrix in Jordan normal form, and denote $M_\mu = M_0 + \mu I$ for every scalar μ. Let B be a matrix with entries in $\mathbb{F}_q$ that is conjugate to*

$$\begin{pmatrix} M_\lambda & & & \\ & M_{F(\lambda)} & & \\ & & \ddots & \\ & & & M_{F^{d-1}(\lambda))} \end{pmatrix}$$

for $d > 1$ such that $\mathbb{F}_q[\lambda] = \mathbb{F}_{q^d}$. Let A be an $n \times n$ matrix with coefficients in $\mathbb{F}_q$. Then,

$$X_B(q, A) = X_{M_\lambda}(q^d, A).$$

Proof. Let $S = \{W \le \mathbb{F}_q^n \mid W \text{ is } A\text{-invariant and } A|_W \sim B\}$ and $T = \{V \le \mathbb{F}_{q^d}^n \mid V \text{ is } A\text{-invariant and } A|_V \sim M_\lambda\}$. We wish to construct a bijection between S and T since $|S| = X_B(q, A)$ and $|T| = X_{M_\lambda}(q^d, A)$.

Consider $W \in S$. Extend scalars to $\mathbb{F}_{q^d}$ to form $\overline{W}$. It is clear that $\overline{W}$ is still A-invariant and A acts by B up to conjugation on $\overline{W}$. Now, consider the function $g : S \to T$ given by sending W to the λ-generalized eigenspace of $A|_{\overline{W}}$. This map produces a subspace that must be A-invariant, as it is a subspace of $\overline{W}$, and on which A acts by M_λ up to conjugation.

Conversely, consider $V \in T$. As V is A-invariant, $Av \in V$ for all $v \in V$, so $F^i(Av) = AF^i(v) \in F^i(V)$ for all i. Thus $F^i(V)$ is A-invariant as well. Furthermore, for any sequence of vectors $v_1, \dots, v_k$ such that $Av_i = \lambda v_i + v_{i-1}$ (with $v_0 = 0$), one has that $A(F(v_i)) = F(A(v_i)) = F(\lambda)F(v_i) + F(v_{i-1})$. Thus, on $F(V)$ the operator A acts through a matrix conjugate to $M_{F(\lambda)}$.

Let $\overline{W} = \bigoplus_{i=0}^{d-1} F^i(V)$. Since every $F^i(V)$ is A-invariant, the sum is also. Furthermore, since $F : F^i(V) \to F^{i+1}(V)$, the sum is also F-invariant. Lastly, A acts on $\overline{W}$ by the block matrix containing the Jordan matrices $M_{F^i(\lambda)}$ for $0 \le i \le d-1$, thus the A action on $\overline{W}$ is conjugate to B by assumption. By Galois descent (see e.g. [2]), there exists a unique $W \le \mathbb{F}_q^n$ whose extension to $\mathbb{F}_{q^d}^n$ is precisely $\overline{W}$. Explicitly, W consists of the F-fixed vectors in $\overline{W}$.

As $\overline{W}$ is A-invariant, for all $w \in W \subset \overline{W}$ we have $Aw \in \overline{W}$. However, we also have $F(Aw) = F(A)F(w) = Aw$ because W consists of the F-fixed vectors. Thus it follows that Aw is F-fixed, and so we must have $Aw \in W$, i.e. W is A-invariant. Also, A acts by B up to conjugation on W because B and $A|_W$ have the same rational canonical form. Thus, $W \in S$, so we have constructed a map from $h : T \to S$ that sends V to W. We now show that g and h are inverses, thus establishing $|S| = |T|$ and completing the proof.

In the construction of $g(h(V)) = g(W)$, one first considers $\overline{W}$ after extending scalars. We then consider the map from $\overline{W}$ to the λ-generalized eigenspace of $A|_{\overline{W}}$, which is V because V is a summand of $\overline{W}$ and on all other summands A has different eigenvalues. We conclude that $g(h(V)) = V$.

Now, consider $h(g(W'))$ for some $W' \in S$, and let $\overline{W'}$ be the space formed by extending the scalars of W' to $\mathbb{F}_{q^d}$. We claim that the $F^i(\lambda)$-generalized eigenspace of $A|_{\overline{W'}}$ is precisely $F^i(g(W'))$. Note that for all $v' \in g(W')$ we have $(A - \lambda I)^m v' = 0$ for some fixed m. Then, $F((A - \lambda I)^m v') = (A - F^i(\lambda)I)^m F^i(v') = 0$, so $F^i(g(W'))$ must be equal to the $F^i(\lambda)$-generalized eigenspace. Thus, we observe that the map h on $g(W')$ precisely takes a direct sum of the eigenspaces of $\overline{W'}$ to produce $\overline{W'}$. Finally, h reduces $\overline{W'}$ back to W' by uniqueness of Galois descent. Therefore, $h(g(W')) = W'$.

This implies g and h are bijections, so $|S| = |T|$. $\square$

The theorem thus proved shows that for B of the above form it suffices to consider statistics X_B where B does indeed have a Jordan normal form in $\mathbb{F}_q$.

3.2 Disjoint Sets of Eigenvalues

We prove that statistics on matrices that have disjoint sets of eigenvalues have simple product expansions. They interact trivially in the following sense.

Proposition 3.2 (Disjoint sets of eigenvalues)**.** *Let A and B be matrices with disjoint sets of eigenvalues. Then the product $X_A \cdot X_B$ is equal to $X_{\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)}$.*

Proof. By Theorem 1.1 there exists coefficients such that,

$$X_A \cdot X_B = \sum_C \lambda^C_{A,B} X_C$$

where C ranges over conjugacy classes of matrices where

$$\max(\dim(A), \dim(B)) \leq \dim(C) \leq \dim(A) + \dim(B).$$

For every such C, the evaluation $X_A(C)$ counts the number of $\dim(A)$-dimensional subspaces of $\mathbb{F}^{\dim(C)}$that are C-invariant and on which C acts by A up to conjugation. Let S_A be the collection of these subspaces, so that $|S_A| = X_A(C)$. Similarly let S_B be the set of the analogous subspaces for the case where A is replaced by B.

Now we claim that for every $V_1 \in S_A$ and $V_2 \in S_B$ we have $V_1 \cap V_2 = \{0\}$. Indeed, the intersection $W = V_1 \cap V_2$ is itself C-invariant, since both V_1 and V_2 are such. Assume for the sake of contradiction that $\dim(W) > 0$. Then the restriction of C to W has some eigenvalue λ (perhaps only a member of a larger field). But since the action of C on V_1 is conjugate to that of A, this λ must then also be an eigenvalue A, and similarly it must also be an eigenvalue of B. This is a contradiction, as A and B have no common eigenvalues.

If the evaluation of C on the product $X_A \cdot X_B$ is non-zero, then both S_A and S_B are non-empty. Thus from the above argument we conclude that for any choice of $V_1 \in S_A$ and $V_2 \in S_B$,

$$\dim(A) + \dim(B) = \dim(V_1) + \dim(V_2) = \dim(V_1 \oplus V_2) \leq \dim(C)$$

showing that the only matrices C for which the product $X_A \cdot X_B$ is non-zero are of the maximal dimension $\dim(A) + \dim(B)$. It also follows that $\mathbb{F}^{\dim(C)} = V_1 \oplus V_2$, that is the space has a basis built from a basis for V_1 followed by a basis for V_2. Since both subspaces are C invariant, in every such basis the matrix C is represented by

$$C' = \left(\begin{smallmatrix} A' & 0 \\ 0 & B' \end{smallmatrix}\right)$$

where $A' = C|_{V_1} \sim A$ and $B' = C|_{V_2} \sim B$, so C' is conjugate to the block matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. But since change of basis corresponds to conjugation, it follows that $C \sim C'$. We thus found that the only matrices of dimension $\leq \dim(A) + \dim(B)$ that may evaluate on $X_A \cdot X_B$ nontrivially must be conjugate to the block matrix built from A and B.

Conversely, this block matrix clearly evaluates to 1 on both X_A and X_B, thus giving the desired equality

$$X_A \cdot X_B = X_{\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)}.$$

□

3.3 Independence of Eigenvalues

The next reduction shows that the choice of eigenvalue for our matrices does not matter. To prove this reduction, we begin with two lemmas.

Lemma 3.3. *A subspace W is A_λ-invariant iff it is $A_{\lambda'}$-invariant.*

Proof. Consider a subspace W that is A_λ-invariant. Note that

$$A_{\lambda'}(W) = (A_\lambda + (\lambda' - \lambda)I)W = A_\lambda(W) + (\lambda' - \lambda)W.$$

Since $A_\lambda(W) \subseteq W$ and $(\lambda' - \lambda)W \subseteq W$, $A_{\lambda'}(W) \subseteq W$. So, W is also $A_{\lambda'}$-invariant. The other direction follows similarly by symmetry. □

Lemma 3.4. *The restriction of A_λ to W is conjugate to B_λ iff the restriction of $A_{\lambda'}$ to W is conjugate to $B_{\lambda'}$.*

Proof. Suppose that the restriction of A_λ to W is conjugate to B_λ. Then, there exists a matrix P such that $A_\lambda|_W = PB_\lambda P^{-1}$. Then,

$$\begin{aligned} A_\lambda|_W P &= PB_\lambda \\ A_\lambda|_W P + ((\lambda' - \lambda)I)P &= PB_\lambda + P((\lambda' - \lambda)I) \\ (A_\lambda + (\lambda' - \lambda)I|_W)P &= P(B_\lambda + (\lambda' - \lambda)I) \\ A_{\lambda'}|_W &= PB_{\lambda'}P^{-1}. \end{aligned}$$

So, the restriction of $A_{\lambda'}$ to W is also conjugate to $B_{\lambda'}$. By symmetry, the backward direction follows, concluding our proof. □

With these lemmas we can now state and proof the proposition.

Proposition 3.5. *Let A_0 and B_0 be nilpotent matrices, and denote $A_\lambda = \lambda I + A_0$ and $A_{\lambda'} = \lambda' I + A_0$. Define B_λ and $B_{\lambda'}$ similarly. Note that A_λ and $A_{\lambda'}$ have the same Jordan form and generalized eigenvectors but different eigenvalues. Then, we have $X_{B_\lambda}(A_\lambda) = X_{B_{\lambda'}}(A_{\lambda'})$.*

Proof. A space W is A_λ-invariant and $A_\lambda|_W \sim B_\lambda$ if and only if W is $A_{\lambda'}$-invariant and $A_{\lambda'}|_W \sim B_{\lambda'}$. Therefore, by definition,

$$X_{B_\lambda}(A_\lambda) = X_{B_{\lambda'}}(A_{\lambda'}).$$

□

Since our statistics are invariant under changing eigenvalues as above, we often work only with unipotent matrices, i.e. ones in which all eigenvalues are 1.

4 Evaluations

In this section, we introduce methods for calculating the polynomials $P_{A,B}(t)$ that determine evaluations of our statistics on unipotent matrices.

We first provide a definition of q-binomial coefficients, which prove useful in our subsequent analysis.

Definition 2. The q-binomial coefficient $\binom{m}{n}_q$ counts the number of n-dimensional subspaces in a m-dimensional space in $\mathbb{F}_q$. Explicitly,

$$\binom{m}{n}_q = \frac{(q^m-1)(q^m-q)(q^m-q^2)\cdots(q^m-q^{n-1})}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})} = \prod_{i=0}^{n-1}\frac{q^m-q^i}{q^n-q^i}$$

if $m \geq n$ and 0 otherwise.

It is a fact that the q-binomial coefficients are polynomials in q (see e.g. [1]).

To begin, we consider the evaluation of statistics associated with matrices composed of Jordan blocks of equal size.

Notation. For $n_1+n_2+\cdots+n_i = n$ let $J_{n_1,n_2,\ldots,n_i}(\lambda)$ denote the $n\times n$ matrix in Jordan form with blocks sizes $n_1, n_2, \ldots, n_i$ and eigenvalue λ. We denote the unipotent matrix $J_{n_1,n_2,\ldots,n_i}(1)$ simply by $J_{n_1,n_2,\ldots,n_i}$.

Lemma 4.1. *Let $B = J_{\underbrace{b_1,\ldots,b_1}_{c_1 \text{ times}}}$ and let $A = J_{a_1,a_2,\ldots,a_k}$. Let t_1 be the largest integer such that $a_{t_1} \geq b_1$. Then,*

$$X_B(A) = \binom{t_1}{c_1}_q \cdot q^{c_1\left[(b_1-1)(t_1-c_1)+\sum_{i=t_1+1}^{k} a_i\right]}$$

Proof. Denote our ambient space by $V = \mathbb{F}_q^{\sum_{i=1}^{k} a_i}$. We need to count the number of A-invariant subspaces $W_1 \leq V$ that have $A|_{W_1} \sim J_{b_1\ldots b_1}$. Setting $N := A - I$ – the nilpotent matrix with the same Jordan blocks as A – the counting problem amounts to computing the cardinality of the set

$$K = \{W_1 < V \mid N(W_1) \subset W_1, N|_{W_1} \sim J_{\underbrace{b_1,\cdots,b_1}_{c_1 \text{ times}}}(0)\}.$$

To do this, consider following quotients:

$$\begin{array}{ccc}
\ker(N^{b_1}) & \subseteq & V \\
\downarrow \pi & & \downarrow \pi \\
\ker(N^{b_1})/\ker(N^{b_1-1}) & \subseteq & V/\ker(N^{b_1-1})
\end{array}$$

We want to establish that the quotient map π induces a surjection from K to the set of c_1-dimensional subspaces of $\ker(N^{b_1})/\ker(N^{b_1-1})$. The next couple of claims suffice: Lemma 4.2 shows that the function is well-defined, and Lemma 4.3 establishes surjectivity.

Lemma 4.2. *For every $W_1 \in K$ we have that $\overline{W}_1 := \pi(W_1)$ is a c_1-dimensional subspace of* $\ker(N^{b_1})/\ker(N^{b_1-1})$

Proof. Fix $W_1 \in K$. Then since $N|_{W_1} \sim J_{\underbrace{b_1,\ldots,b_1}_{c_1 \text{ times}}}(0)$ it follows that W_1 has a basis of the form

$$\begin{aligned}
&\{v_1, Nv_1, N^2v_1, \ldots, N^{b_1-1}v_1\\
&v_2, Nv_2, N^2v_2, \ldots, N^{b_1-1}v_2\\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots\\
&v_{c_1}, Nv_{c_1}, N^2v_{c_1}, \ldots, N^{b_1-1}v_{c_1}\}
\end{aligned}$$

such that $N^{b_1}v_i = 0$ for all i. Thus $\overline{W}_1 \leq \ker(N^{b_1})$.

Furthermore, note that $N^{b_1-1}(Nv_i) = N^{b_1}v_i = 0$ so all $N^j v_i$ project to 0 for $j \geq 1$ under π. We conclude that $\overline{W}_1 = \pi(W_1)$ is spanned by $\langle \pi(v_1), \pi(v_2), ..., \pi(v_{c_1}) \rangle$ and is of dimension at most c_1. We show that $\pi(v_i)$ are indeed independent. Suppose

$$0 = \lambda_1\pi(v_1) + \lambda_2\pi(v_2) + ... + \lambda_{c_1}\pi(v_{c_1})$$

then,

$$0 = \pi(\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_{c_1} v_{c_1}).$$

Thus, we know that $\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_{c_1} v_{c_1} \in \ker(N^{b_1-1})$. So,

$$\begin{aligned}
0 &= N^{b_1-1}(\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_{c_1} v_{c_1})\\
&= \lambda_1 N^{b_1-1}v_1 + \lambda_2 N^{b_1-1}v_2 + \cdots + \lambda_{c_1} N^{b_1-1}v_{c_1}.
\end{aligned}$$

But since $\{N^{b_1-1}v_1, N^{b_1-1}v_2, \ldots, N^{b_1-1}v_{c_1}\}$ are members of a basis of W, we conclude $\lambda_1 = \lambda_2 = \cdots = \lambda_{c_1} = 0$. Thus $\pi(v_1), \pi(v_2), ...\pi(v_{c_1})$ are linearly independent and $\overline{W}_1$ is of dimension c_1. □

Lemma 4.3. *For all c_1-dimensional subspaces $\overline{W}_1 \leq \ker(N^{b_1})/\ker(N^{b_1-1})$ there exists $W_1 \in K$ such that $\pi(W_1) = \overline{W}_1$*

Proof. Let $\overline{W}_1$ be a c_1-dimensional space in $\ker(N^{b_1})/\ker(N^{b_1-1})$ and pick $\{v_1, \ldots, v_{c_i}\} \subset \ker(N^{b_1})$ such that $\{\pi(v_1), \pi(v_2), \ldots, \pi(v_{c_1})\}$ form a basis for W. Consider

$$\begin{aligned}
W_1 = \operatorname{span}(&v_1, Nv_1, N^2v_1, \ldots, N^{b_1-1}v_1,\\
&v_2, Nv_2, N^2v_2, \ldots, N^{b_1-1}v_2,\\
&\vdots \qquad \vdots \qquad \vdots\\
&v_{c_1}, Nv_{c_1}, N^2v_{c_1}, \ldots, N^{b_1-1}v_{c_1}).
\end{aligned}$$

We claim that $W_1 \in K$ and $\pi(W_1) = \overline{W}_1$.

Since $N^j v_i \in \ker(N^{b_1-1})$ for $j \geq 1$ vanishes in the quotient,

$$\begin{aligned}
\pi(W_1) = \langle &\pi(v_1), 0, 0, \ldots, 0\\
&\pi(v_2), 0, 0, \ldots, 0\\
&\vdots \qquad \vdots \qquad \vdots\\
&\pi(v_{c_1}), 0, 0, \ldots, 0\rangle\\
= \langle &\pi(v_1), \pi(v_2), \ldots, \pi(v_{c_1})\rangle = \overline{W}_1.
\end{aligned}$$

To see that $W_1 \in K$ we must show that $N(W_1) \subseteq W_1$ and $N|_{W_1} \sim J_{\underbrace{b_1,\cdots,b_1}_{c_1 \text{ times}}}(0)$. The N-invariance is clear on the spanning set defining W_1, so the resulting span W_1 must be N-invariant as well.

To determine the conjugacy class of $N|_{W_1}$, we only need to show that the spanning set is linearly independent, since this would imply that $\{N^j v_i\}_{i,j}$ is a basis with respect to which $N|_{W_1}$ has the desired Jordan form.

Suppose that

$$\begin{array}{l} \lambda_{1,1}v_1 + \lambda_{1,2}Nv_1 + \lambda_{1,3}N^2v_1 + \cdots + \lambda_{1,b_1}N^{b_1-1}v_1 \\ +\lambda_{2,1}v_2 + \lambda_{2,2}Nv_2 + \lambda_{2,3}N^2v_2 + \cdots + \lambda_{2,b_1}N^{b_1-1}v_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ +\lambda_{c_1,1}v_{c_1} + \lambda_{c_1,2}Nv_{c_1} + \lambda_{c_1,3}N^2v_{c_1}, \cdots + \lambda_{c_1,b_1}N^{b_1-1}v_{c_1} = 0. \end{array}$$

Applying π to this equation to get

$$\lambda_{1,1}\pi(v_1) + \lambda_{2,1}\pi(v_2) + \lambda_{3,1}\pi(v_3) + \ldots \lambda_{c_1,1}\pi(v_{c_1}) = 0.$$

But $\{\pi(v_1), \pi(v_2), \ldots, \pi(v_{c_1})\}$ was chosen to be a basis for $\overline{W}_1$ so $\lambda_{1,1} = \lambda_{2,1} = \cdots = \lambda_{c_1,1} = 0$. We continue this process by induction on j, the column number in the equation above.

Assume that all $\lambda_{i,k} = 0$ for all i and for $k < j$. So the linear relation reduces to

$$N^j \left(\sum_{i=1}^{c_1} \sum_{k=j}^{b_1-1} \lambda_{i,k} N^{k-j} v_i \right) = 0$$

which implies that the argument $\sum \lambda_{i,k} N^{k-j} v_i$ already vanishes in the quotient. Applying π to the argument, we get

$$\lambda_{1,j}\pi(v_1) + \ldots + \lambda_{c_1,j}\pi(v_{c_1}) = 0$$

which again by linear independence of the $\pi(v_i)$'s gives $\lambda_{1,j} = \ldots = \lambda_{c_1,j} = 0$, thus completing the induction step.

We conclude that $\{N^j v_i\}_{i,j}$ is indeed a basis, so $W_1 \in K$ and $\pi(W_1) = \overline{W}_1$, thus the proof is complete. □

Recall that t_1 is the number of blocks in A and N of size at least b_1. Then we have

$$\dim(\ker N^{b_1}) = (b_1)(t_1) + \sum_{i=t_1+1}^{k} a_i,$$
$$\dim(\ker N^{b_1-1}) = (b_1 - 1)(t_1) + \sum_{i=t_1+1}^{k} a_i,$$

thus there are

$$\binom{\dim(\ker N^{b_1}) - \dim(\ker N^{b_1-1})}{c_1}_q = \binom{t_1}{c_1}_q$$

ways to pick a c_1-dimensional subspace $\overline{W}_1 \leq \ker(N^{b_1})/\ker(N^{b_1-1})$.

So we have constructed a function from K to a set of size $\binom{t_1}{c_1}_q$. It remains to count the number of preimages each $\overline{W_1}$ has in K.

Lemma 4.4. *Fix a c_1-dimensional subspace $\overline{W}_1 \leq \ker(N^{b_1})/\ker(N^{b_1-1})$. There are precisely*

$$q^{c_1\left[(b_1-1)(t_1-c_1)+\sum_{i=t_1+1}^{k} a_i\right]}$$

preimages $W_1 \in K$ such that $\pi(W_1) = \overline{W}_1$.

Proof. If $s : \overline{W}_1 \to \ker(N^{b_1})$ is any section of π, with image W_0, then the proof of Lemma 4.3 shows that $W_1 := \sum_j N^j(W_0)$ satisfies $W_1 \in K$ and $\pi(W_1) = \overline{W}_1$. So we begin by counting sections of $\pi : \pi^{-1}(\overline{W}_1) \to \overline{W}_1$.

Fix a section $s_0 : \overline{W}_1 \to W_0$. Then the set of sections is in bijection with

$$\hom(\overline{W}_1, \ker(N^{b_1-1}))$$

where a homomorphism $\phi : \overline{W} \to \ker(N^{b_1-1})$ corresponds to the section $s_0 + \phi$. It follows that the number of sections is

$$q^{\dim \overline{W}_1 \cdot \dim \ker(N^{b_1-1})} = q^{c_1((b_1-1)t_1 + \sum_{i>t_1} a_i)}.$$

However, in going from sections to preimages $W_1 \in K$ there is some overcounting, which we now address. Fix some $W_1 \in K$ such that $\pi(W_1) = \overline{W}_1$, we wish to count the number of sections of $\pi : W_1 \to \overline{W}_1$. The same argument as in the previous paragraph shows that the number such sections is

$$|\hom(\overline{W}_1, W_1 \cap \ker(N^{b_1-1}))| = q^{c_1 \cdot \dim W \cap \ker(N^{b_1-1})}.$$

But because $N|_{W_1} \sim J_{b_1,\ldots,b_1}(0)$, it follows that $\dim(W_1 \cap \ker(N^{b_1-1})) = (b_1-1)c_1$. This is the overcounting involved in counting sections, so we divide by this amount to get the number of preimages:

$$q^{c_1(b_1-1)t_1 + \sum_{i>t_1} a_i - c_1(b_1-1)c_1}$$

as claimed. ☐

Combining all claims in the proof, the set K of interest maps onto a set of size $\binom{t_1}{c_1}_q$ with fibers of equal size given by the previous claim. It follows that

$$X_B(A) = |K| = \binom{t_1}{c_1}_q \cdot q^{c_1\left[(b_1-1)(t_1-c_1) + \sum\limits_{i=t_1+1}^{k} a_i\right]}.$$

This completes the proof of Lemma 4.1. ☐

With Lemma 4.1, we now prove the general case by induction.

Theorem 4.5 (General Evaluation Formula)**.** *Let*

$$B = J_{\underbrace{b_1,\ldots,b_1}_{c_1 \text{ times}},\underbrace{b_2,\ldots,b_2}_{c_2 \text{ times}},\ldots,\underbrace{b_n,b_n,\ldots,b_n}_{c_n \text{ times}}}$$

where $b_1 > b_2 > \cdots > b_n$ and there are c_i Jordan blocks of size b_i in B. Let $A = J_{a_1,a_2,\ldots,a_k}$ where $a_1 \geq a_2 \geq \cdots \geq a_k$. Let t_i be the number of blocks of A of size at least b_i, i.e. the largest integer such that $a_{t_i} \geq b_i$. Then,

$$X_B(A) = \left(\prod_{i=1}^{n} \binom{t_i - \sum\limits_{j=1}^{i-1} c_j}{c_i}_q\right) \cdot q^{\sum\limits_{i=1}^{n} c_i\left((b_i-1)(t_i+c_i-2\sum\limits_{j=1}^{i} c_j) - \sum\limits_{j=i+1}^{n} c_j + \sum\limits_{j=t_i+1}^{k} a_j\right)}.$$

Proof. We prove this theorem by induction on the number of distinct block sizes in B.

The base case is when there are 0 blocks in B. This corresponds to the empty matrix on the zero vector space. The only vector space counted is the zero space, so the count is 1. This is precisely the right hand side of the formula, $q^0 = 1$.

Assume by induction that the formula is true for matrices with n distinct Jordan block sizes. We will prove the formula holds for matrices with $n+1$ distinct block sizes. Let b_1 be the largest block size and c_1 its multiplicity.

As in the proof of Lemma 4.1, we set $N := A - I$ for the nilpotent matrix. Now, since $(B-I)^{b_1} = 0$ we note that any A-invariant space W on which $A|_W \sim B$ has $N^{b_1}(W) = (A-I)^{b_1}(W) = 0$. So for the purpose of counting subspaces W of this form, it is sufficient to restrict the ambient space to $\ker(N^{b_1})$, which we denote by V. On this smaller ambient space, the transformations A and N have their Jordan blocks restricted to have size at most b_1. Thus, without loss of generality we assume that $b_1 \geq a_i$ for all i.

By Lemma 4.1, we know that the number of A-invariant subspaces on which A acts by a transformation conjugate to $J_{\underbrace{b_1,\ldots,b_1}_{c_1 \text{ times}}}$ is

$$\binom{t_1}{c_1}_q \cdot q^{c_1\left[(b_1-1)(t_1-c_1)+\sum_{j=t_1+1}^{k} a_j\right]}. \tag{3}$$

Let W_1 be any subspace of this form. We wish to count the number of way to extend W_1 in an A-invariant way so that A acts by a transformation conjugate to B. This is equivalent to finding a subspace of V/W_1 that is A-invariant and on which A is conjugate to $J_{\underbrace{b_2,\ldots,b_2}_{c_2 \text{ times}},\ldots,\underbrace{b_{n+1},b_{n+1},\ldots,b_{n+1}}_{c_{n+1} \text{ times}}}$. Since this latter matrix has n distinct Jordan block sizes, our inductive hypothesis applies, and the number of such subspaces is known. However, note that the transformation induced by A on the quotient, call it A', is represented by a matrix obtained from A by removing c_1 blocks of the maximal size b_1. This means that for A' we have $t'_i = t_i - c_1$ for all $i \geq 2$.

Therefore, the number of ways to pick the subspace $W' \leq V/W_1$ is

$$\left(\prod_{i=2}^{n+1} \binom{t_i - c_1 - \sum_{j=2}^{i-1} c_i}{c_i}_q\right) \cdot q^{\sum_{i=2}^{n} c_i\left((b_i-1)(t_i-c_1+c_i-2\sum_{j=2}^{i} c_j)-\sum_{j=i+1}^{n} c_j+\sum_{j=t_i+1}^{k} a_j\right)}. \tag{4}$$

The preimage W such that $W/W_1 = W'$ is uniquely determined by W', and is an A-invariant subspace satisfying $A|_W \sim B$.

We thus counted the number of pairs (W_1, W) such that $W_1 \leq W$ are two A-invariant spaces with respective restrictions of A conjugate to $J_{b_1,\ldots,b_1}$ and B. However, we are interested in counting only the set of subspaces W, so we must divide by the number of pairs (W_1, W) with a given W.

Fixing W, the number of choices for $W_1 \leq W$ that is A-invariant and on which A acts as $J_{b_1,\ldots,b_1}$ is again counted in Lemma 4.1. Since A acts on W as the matrix B we have $t_i = c_i$ for all i and $a_{t_i} = b_i$, thus the lemma gives the number of such $W_1 \leq W$ to be

$$\binom{c_1}{c_1}_q \cdot q^{c_1(b_1-1)(c_1-c_1)+\sum_{i\geq 2} c_i b_i} = 1 \cdot q^{c_1 \sum_{i \geq 2} c_i b_i}.$$

Dividing the product of (3) and (4) by this overcounting factor, we get the number of desired spaces W to be:

$$\left(\prod_{i=1}^{n+1} \binom{t_i - \sum_{j=1}^{i-1} c_i}{c_i}_q\right) \cdot q^{f(a,b,c,t)}$$

where the exponent is

$$
\begin{aligned}
f(a,b,c,t) &= c_1\left[(b_1-1)(t_1-c_1)+\sum_{j=t_1+1}^{k} a_j\right] \\
&+ \sum_{i=2}^{n} c_i\left((b_i-1)(t_i-c_1+c_i-2\sum_{j=2}^{i} c_j)-\sum_{j=i+1}^{n+1} c_j+\sum_{j=t_i+1}^{k} a_j\right)-\sum_{i=2}^{n+1} c_1c_ib_i \\
&= \sum_{i=1}^{n+1} c_i\left((b_i-1)(t_i+c_i-2\sum_{j=1}^{i} c_j)+\sum_{j=t_i+1}^{k} a_j\right) \\
&\quad +\sum_{i=2}^{n+1} c_i\left((b_i-1)(c_1)-\sum_{j=i+1}^{n+1} c_j-c_1b_i\right) \\
&= \sum_{i=1}^{n+1} c_i\left((b_i-1)(t_i+c_i-2\sum_{j=1}^{i} c_j)+\sum_{j=t_i+1}^{k} a_j-\sum_{j=i+1}^{n+1} c_j\right)
\end{aligned}
$$

thus completing the proof of the induction step. □

We now have the tools to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Theorem 4.5 shows that for every two unipotent Jordan matrices A and B there exists a polynomial $P_{A,B}(t)$ such that $X_B(A) = P_{A,B}(q)$ when considered as matrices over any finite field $\mathbb{F}_q$. Furthermore the polynomial in Theorem 4.5 depends only on the block sizes of A and B, i.e. by the partitions describing the block sizes. If λ and μ are the partitions enumerating the block sizes of A and B respectively, we write $P_{\lambda,\mu}(t)$ for the resulting polynomial. □

Proof of Theorem 1.3. To show that there exists a polynomial $P_{A,B}(t)$ for any evaluation $X_B(A)$ for inveritble matrices A and B, we use the results in Section 3.

First, let $f_B(t)$ be the characteristic polynomial of B and factor f_B over $\mathbb{F}_q[x]$ into distinct irreducible polynomials $f_1, f_2, \dots, f_k$ with multiplicities $m_1, \dots, m_k$. That is,

$$f_B(t) = f_1(t)^{m_1} f_2(t)^{m_2} \cdots f_k(t)^{m_k}.$$

Let B_i the restriction of B to the subspace $\ker(f_i(B)^{m_i})$. Observe that the characteristic polynomial of B_i is $f_i(t)^{m_i}$ and $B \sim \bigoplus_{i=1}^{k} B_i$.

Since each pair of the B_i have coprime characteristic polynomials, they have disjoint sets of eigenvalues. So by Proposition 3.2, we have $X_B(A) = \prod_{i=1}^{k} X_{B_i}(A)$. The characteristic polynomial of B_i can be expressed as $f_{B_i}(t) = (t-\lambda_i)^{n_i}(t-F(\lambda_i))^{n_i}\cdots(t-F^{r_i-1}(\lambda_i))^{n_i}$ for some n_i and r_i where $\lambda_i \in \mathbb{F}_{q^{r_i}}$. Furthermore, $B_i \sim \bigoplus_{j=0}^{r_i-1} F^j(M_{\lambda_i})$ where M_{λ_i} is an $n_i \times n_i$ matrix in Jordan normal form with eigenvalue λ_i. Therefore, by Theorem 3.1, $X_{B_i}(A) = X_{M_{\lambda_i}}(q^{r_i}, A)$.

Let $A = N_{\lambda_i} \oplus N'$ where N_{λ_i} is in Jordan form over $\mathbb{F}_{q^{r_i}}$ and has eigenvalue λ_i and N' has eigenvalues distinct from λ_i. This decomposition is possible since $\lambda_i \in \mathbb{F}_{q^{r_i}}$. We claim that $X_{M_{\lambda_i}}(q^{r_i}, N_{\lambda_i} \oplus N') = X_{M_{\lambda_i}}(q^{r_i}, N_{\lambda_i})$. Indeed, consider a subspace W counted by $X_{M_{\lambda_i}}(q^{r_i}, A)$, i.e. A-invariant W such that $A|_W \sim M_{\lambda_i}$. The only generalized eigenspace of $A|_W$ has eigenvalue λ_i and is thus contained in the domain of N_{λ_i}.

Let μ_i and λ_i be the partitions that represent the Jordan block sizes of M_{λ_i} and N_{λ_i}, respectively. By Proposition 3.5, $X_{M_{\lambda_i}}(q^{r_i}, N_{\lambda_i}) = P_{\lambda_i,\mu_i}(q^{r_i})$. Thus,

$$X_B(A) = \prod_{i=1}^{k} X_{B_i}(A) = \prod_{i=1}^{k} P_{\lambda_i,\mu_i}(q^{r_i}) \tag{5}$$

as claimed. ☐

The evaluation formula has a surprising consequence that the size of the largest Jordan block does not matter as long as its multiplicity is the same in A and B.

Corollary 4.6. *Suppose that A and B are unipotent Jordan matrices such that the largest Jordan block of B has size b. If there are c Jordan blocks in B of size b and c Jordan blocks in A of size at least b, then $X_B(A)$ is independent of b.*

Proof. This is immediate from Theorem 4.5 as $t_1 = c_1$, so the coefficient of b in the exponent is 0. ☐

5 Product Expansions

Now that we have determined a precise formula for the evaluation of any given statistic at any matrix, we can combine this with our aforementioned algorithm to compute some of our sought-after expansion coefficients.

As with evaluations, it is sufficient to consider expansion coefficients for unipotent Jordan matrices because the product $X_A \cdot X_B$ evaluated at each matrix C can be written as a product of polynomials $P_{\lambda,\mu}$ for partitions λ and μ that represent the respective Jordan block sizes of unipotent matrices.

We first discuss some general results for all product expansions for unipotent Jordan matrices before discussing a few specific cases.

5.1 General Results

Since all the expansion coefficients for unipotent Jordan matrices are polynomials as per Theorem 4.5, we can also show that the expansion coefficients of these statistics are polynomials.

Proof of Theorem 1.4. This corollary follows inductively by the algorithm described in Section 2. Let A and B have dimensions n_1 and n_2, respectively. Let all of the conjugacy classes of matrices with dimensions between $\max(n_1, n_2)$ and $n_1 + n_2$ be $C_1, C_2, \ldots, C_k$ such that $\dim C_1 \leq \dim C_2 \leq \cdots \leq \dim C_k$. We show by induction that the coefficient of X_{C_i} is a polynomial in q.

The base case is $i = 1$. By Equation (2), the coefficient of X_{C_1} is

$$\lambda_{A,B}^{C_1} = X_A(C_1) \cdot X_B(C_1)$$

which is a polynomial in q due to Theorem 1.3. Assume by induction that for $i \leq m$, the coefficient of X_{C_i} is a polynomial in q. Then, the coefficient of $X_{C_{m+1}}$ is

$$\lambda_{A,B}^{C_{m+1}} = X_A(C_{m+1}) \cdot X_B(C_{m+1}) - \sum_{i=1}^{m} \lambda_{A,B}^{C_i} X_{C_i}(C_{m+1}).$$

By our inductive hypothesis and Theorem 1.3, all terms on the right-hand side are polynomials in q, so the coefficent of $X_{C_{m+1}}$ is likewise a polynomial in q. ☐

Many of the coefficients in the aforementioned product expansion are, in fact, the zero polynomial. The necessary condition for when the coefficient is a nonzero polynomial is given by Proposition 1.5, which we prove with Algorithm 2 and Theorem 4.5.

Proof of Proposition 1.5. It suffices to show that $\lambda_{A,B}^{C} = 0$ whenever the maximum Jordan block of C is not equal to b. We prove this with induction. Let all of the conjugacy classes of matrices with dimensions between 1 and $n_1 + n_2$, inclusive, be $C_1, C_2, \ldots, C_k$ such that $\dim C_1 \leq \dim C_2 \leq$

$\cdots \leq \dim C_k$. Let j be the smallest positive integer such that X_{C_j} has a nonzero coefficient in the expansion of $X_A \cdot X_B$. By minimality of j and Equation (2),

$$\lambda_{A,B}^{C_j} = X_A(C_j)X_B(C_j) - \sum_{i=1}^{j} \lambda_{A,B}^{C_i} X_{C_i}(C_j) = X_A(C_j)X_B(C_j).$$

Now, if the maximum Jordan block size in C_j is less than b, then $X_B(C_j) = 0$ from Theorem 4.5. But this is a contradiction since it would yield $\lambda_{A,B}^{C_j} = 0$. On the other hand, if $C_j = J_{b+t_1,b+t_2,\ldots,b+t_s,c_{j_1},c_{j_2},\ldots,c_{j_k}}$ for $t_1,\ldots,t_s > 0$, let $C_j' = J_{\underbrace{b,b,\ldots,b}_{s \text{ times}},c_{j_1},c_{j_2},\ldots,c_{j_k}}$. Then, Theorem 4.5 yields

$$\begin{aligned}
\lambda_{A,B}^{C_j} = X_A(C_j)X_B(C_j) &= X_A(C_j')X_B(C_j') \\
&= \lambda_{A,B}^{C_j'} + \sum_{\dim M < \dim C_j - t} \lambda_{A,B}^{M} X_M(C_j') \\
&= \lambda_{A,B}^{C_j'}.
\end{aligned}$$

This implies that the coefficient of $X_{C_j'}$ is nonzero, which contradicts the minimality of j since $\dim(C_j') < \dim(C_j)$.

This is the base case for our induction. Assume by induction that for all $i \leq m$, the coefficient of X_{C_i} is 0 when the largest Jordan block of C_i is not equal to b, then we will prove the same holds for $m+1$. From Equation (2),

$$\lambda_{A,B}^{C_{m+1}} = X_A(C_{m+1}) \cdot X_B(C_{m+1}) - \sum_{i=1}^{m} \lambda_{A,B}^{C_i} X_{C_i}(C_{m+1}).$$

If the largest Jordan block of C_{m+1} is less than b, then $X_B(C_{m+1}) = 0$ by Theorem 4.5. Similarly, for every nonzero $\lambda_{A,B}^{C_i}$ in the sum, C_i has largest Jordan block b by the induction hypothesis, so $X_{C_i}(C_{m+1}) = 0$. Therefore, we have that $\lambda_{A,B}^{C_{m+1}} = 0$.

Now, instead suppose $C_{m+1} = J_{b+t_1,b+t_2,\ldots,b+t_s,c_1,c_2,\ldots,c_k}$ for some $t_1,\ldots,t_s > 0$. Let $C_\ell = J_{\underbrace{b,b,\ldots,b}_{s \text{ times}},c_1,c_2,\ldots,c_k}$ where $\ell < m+1$. Thus,

$$\begin{aligned}
\lambda_{A,B}^{C_{m+1}} &= X_A(C_{m+1})X_B(C_{m+1}) - \sum_{i=1}^{m} \lambda_{A,B}^{C_i} X_{C_i}(C_{m+1}) \\
&= X_A(C_{m+1})X_B(C_{m+1}) - \sum_{i=1}^{\ell-1} \lambda_{A,B}^{C_i} X_{C_i}(C_{m+1}) - \lambda_{A,B}^{C_\ell} X_{C_\ell}(C_{m+1}) \\
&\qquad - \sum_{i=\ell+1}^{m} \lambda_{A,B}^{C_i} X_{C_i}(C_{m+1}).
\end{aligned}$$

By Theorem 4.5, $X_A(C_{m+1}) = X_A(C_\ell)$ and $X_B(C_{m+1}) = X_B(C_\ell)$. We also know $X_{C_\ell}(C_{m+1}) = 1$. For $1 \leq i \leq \ell - 1$ where $\lambda_{A,B}^{C_i}$ is nonzero, $X_{C_i}(C_{m+1}) = X_{C_i}(C_\ell)$ by the inductive hypothesis since the largest Jordan block of C_i is b. Furthermore, for all i such that $\ell + 1 \leq i \leq m$, whenever $\lambda_{A,B}^{C_i}$ is nonzero, the largest Jordan block of C_i is b by the inductive hypothesis. In this case, by Proposition 2.1, $X_{C_i}(C_{m+1}) = X_{C_i}(C_\ell) = 0$ since $\dim(C_\ell) \leq \dim(C_i)$ and $C_i \neq C_\ell$. Therefore, we have,

$$\lambda_{A,B}^{C_{m+1}} = X_A(C_\ell)X_B(C_\ell) - \sum_{i=1}^{\ell-1} \lambda_{A,B}^{C_i} X_{C_i}(C_\ell) - \lambda_{A,B}^{C_\ell} = 0.$$

by Equation (2) applied to $\lambda^{C_\ell}_{A,B}$. This completes the induction step and we conclude that the coefficient $\lambda^C_{A,B}$ is only nonzero if the largest Jordan block of C is b. □

The condition provided in Proposition 1.5 greatly reduces our calculations with the algorithm described in Section 2 in the specific cases below.

5.2 Single Jordan Blocks

We find the following expansion for the product of statistics on single Jordan blocks.

Theorem 5.1. *For $b \geq a$, we have the following expansion:*

$$X_{J_a} \cdot X_{J_b} = \begin{cases} X_{J_b} + \left(\sum_{m=1}^{a-1} q^{2m-1}(q-1)X_{J_{b,m}}\right) + q^{2a}X_{J_{b,a}} & b > a \\ X_{J_b} + \left(\sum_{m=1}^{a-1} q^{2m-1}(q-1)X_{J_{b,m}}\right) + q^{2a-1}(q+1)X_{J_{b,a}} & b = a \end{cases} \tag{6}$$

Proof. We only prove the case when $b > a$ as the other case is similar.

We can find the coefficients in Equation (6) recursively using the algorithm in Section 2. However, since there exists an expansion

$$X_{J_a} \cdot X_{J_b} = \sum_C \lambda^C_{J_a,J_b} X_C$$

from Theorem 1.1, we simply verify that Equation (6) holds for all the matrices C such that X_C may have nonzero coefficient. From Proposition 1.5, C must be of the form $J_{b,a_1,a_2,\ldots,a_n}$. For $1 \leq m \leq a$, define t_m to be the largest integer $1 \leq i \leq n$ such that $a_i \geq m$. If $a_i < m$ for all i, let $t_m = 0$. Note that $t_1 = n$. Furthermore, let $k = \sum_{i=1}^n a_i$.

First, for $C = J_{b,a_1,a_2,\ldots,a_n}$, observe that $\sum_{i=1}^n a_i \leq a < b$, so each a_i is less than b. Therefore, note that $X_{J_b}(C) = q^k$ from Theorem 4.5. Now, consider the evaluation of $X_{J_{b,m}}(C)$. By Theorem 4.5,

$$X_{J_{b,m}}(C) = \binom{t_m}{1}_q \cdot q^{(m-1)(t_m-2)+k-1+\sum_{j=t_m+1}^n a_j}$$

Consider the sum in the exponent. We can evaluate it directly based on the sequence $\{t_i\}$. Since there are $t_j - t_{j+1}$ terms in the sequence $\{a_i\}$ that are equal to j, we have

$$\sum_{j=t_m+1}^{n} a_j = \sum_{j=1}^{m-1}(t_j - t_{j+1})j = -(m-1)t_m + \sum_{j=1}^{m-1} t_j.$$

Therefore, the exponent is

$$(m-1)t_m - 2m + 2 + k - 1 - (m-1)t_m + \sum_{j=1}^{m-1} t_j = -2m + 1 + k + \sum_{j=1}^{m-1} t_j.$$

Our evaluation becomes $X_{J_{b,m}}(C) = \binom{t_m}{1}_q \cdot q^{-2m+1+k+\sum_{j=1}^{m-1} t_j}$. Finally, note that $X_{J_a}(C) = \binom{t_a+1}{1}_q q^{(a-1)t_a+\sum_{j=t_a+1}^n a_j} = \binom{t_a+1}{1}_q q^{\sum_{j=1}^{a-1} t_j}$.

The RHS of Eq. (6) evaluates on C as

$$
\begin{aligned}
&X_{J_b}(C)+\left(\sum_{m=1}^{a-1} q^{2m-1}(q-1)X_{J_{b,m}}(C)\right)+q^{2a}X_{J_{b,a}}(C)\\
&=q^k+\sum_{m=1}^{a-1} q^{2m-1}(q-1)\binom{t_m}{1}_q\cdot q^{-2m+1+k+\sum_{j=1}^{m-1}t_j}+q^{2a}\binom{t_a}{1}_q\cdot q^{-2a+1+k+\sum_{j=1}^{a-1}t_j}\\
&=q^k+\sum_{m=1}^{a-1}(q^{t_m}-1)q^{k+\sum_{j=1}^{m-1}t_j}+\binom{t_a}{1}_q q^{k-1+\sum_{j=1}^{a-1}t_j}\\
&=q^k+\sum_{m=1}^{a-1}(q^{k+\sum_{j=1}^{m}t_j}-q^{k+\sum_{j=1}^{m-1}t_j})+\frac{q^{k+1+\sum_{j=1}^{a}t_j}-q^{k+1+\sum_{j=1}^{a-1}t_j}}{q-1}\\
&=q^k-q^k+q^{k+\sum_{j=1}^{a-1}t_j}+\frac{q^{k+1+\sum_{j=1}^{a}t_j}-q^{k+1+\sum_{j=1}^{a-1}t_j}}{q-1}=\binom{t_a+1}{1}_q q^{k+\sum_{j=1}^{a-1}t_j}.
\end{aligned}
$$

One can check that this is equal to $X_{J_a}(C)X_{J_b}(C)$ from our calculations above, so there is no remaining non-trivial linear contribution for X_C, and Eq. (6) is satisfied by all conjugacy classes C. □

Remark 5.1.1. In the expansion $X_{J_b}\cdot X_{J_a}$, where $b\ge a$, the only non-trivial contributions to the expansion are from conjugacy classes with one or two Jordan blocks.

Corollary 5.2. *All statistics $X_{J_{b,a}}$ can be expressed as degree 2 polynomials in $\{X_{J_n}|n\in\mathbb{N}\}$ with coefficients in rational functions $\mathbb{Q}(q)$.*

Proof. We fix b and induct on $a\le b$. The base case $a=1$ follows as $X_{J_{b,1}}=\frac{X_{J_b}\cdot X_{J_1}-X_{J_b}}{q^2}$ from Theorem 5.1. Assume by induction that all $X_{J_{b,m}}$ for $1\le m\le k<b$ are given by degree 2 polynomials in the X_{J_n}'s. When $k+1<b$, Theorem 5.1 yields

$$X_{J_{b,k+1}}=\frac{X_{J_b}\cdot X_{J_k}-X_{J_b}-\left(\sum_{m=1}^{k} q^{2m-1}(q-1)X_{J_{b,m}}\right)}{q^{2k+2}}$$

As each term on the RHS can be is expressed as a degree 2 polynomial in the X_{J_n}'s, we have that $X_{b,k+1}$ is similarly expressed in terms of the X_{J_n}'s. When $k+1=b$, Theorem 5.1 similarly yields

$$X_{J_{b,k+1}}=\frac{X_{J_b}^2-X_{J_b}-\left(\sum_{m=1}^{b-1} q^{2m-1}(q-1)X_{J_{b,m}}\right)}{q^{2b-1}(q+1)}.$$

For the same reason as before, this can be expressed in terms of the X_{J_n}'s, and we are done. □

We note that the expansion coefficients in Theorem 5.1 are the same for every b once $b\ge a+1$: in other words, they depend solely on a. It seems reasonable to conjecture that such a phenomenon extends to a more general case.

Conjecture 5.3. *Given $a_1\ge b_1$, $a_1\ge a_2\ge\cdots\ge a_k$, and $b_1\ge b_2\ge\cdots\ge b_j$, the product $X_{J_{a_1,\ldots,a_k}}\cdot X_{J_{b_1,b_2,\ldots,b_j}}$ has the same expansion coefficients when a_i is increased for any $1\le i\le k$ such that $a_i>b_1$.*

5.3 Larger Jordan Matrices

Using similar methods as before, we can try to determine the expansion coefficients for the product of a statistic on a single Jordan block with a statistic on two Jordan blocks.

An interesting example can thus be proven:

Lemma 5.4. *When $b > a > 1$,*

$$\begin{aligned} X_{J_b} \cdot X_{J_{a,1}} &= qX_{J_{b,1}} + \sum_{m=2}^{a-1} q^{2m-2}(q-1)X_{J_{b,m}} + q^{2a-1}X_{J_{b,a}} \\ &+ q^2(q^2-1)X_{J_{b,1,1}} + \sum_{m=2}^{a-1} q^{2m+1}(q-1)X_{J_{b,m,1}} + q^{2a+2}X_{J_{b,a,1}}. \end{aligned} \tag{7}$$

Proof. By Proposition 1.5, the product expansion of $X_{J_b} \cdot X_{J_{a,1}}$ contains only terms $\lambda^C_{J_b,J_{a,1}} X_C$ where C is of the form $J_{b,a_1,a_2,\dots,a_n}$ with $a_1 \geq a_2 \geq \cdots \geq a_n$. For $1 \leq m \leq a$, define t_m to be the largest integer $1 \leq i \leq n$ such that $a_i \geq m$. If $a_i < m$ for all i, let $t_m = 0$. Note that $t_1 = n$. We also denote $s_i = \sum_{j=1}^{i} t_j$ and $k = \sum_{j=1}^{n} a_j$.

As before, the expansion coefficients can be determined recursively, but we simply verify that Theorem 5.4 is satisfied for all matrices of the form $J_{b,a_1,a_2,\dots,a_n}$. First, observe that the only matrices in which $t_a > 0$ are $J_{b,a}$, $J_{b,a,1}$ and $J_{b,a+1}$. We can check these cases manually.

For all other matrices, we have $t_a = 0$. To show that the equation above is true, we evaluate every term at the matrix C. We find the evaluations on the left hand side first:

$$X_{J_b}(C) = q^k,$$

$$\begin{aligned} X_{J_{a,1}}(C) &= \binom{t_a+1}{1}_q \binom{t_1}{1}_q \cdot q^{-1+(a-1)t_a+\sum_{j=t_a+1}^{n} a_j} \\ &= \frac{q^{n-1}-1}{q-1} \cdot q^{-1+\sum_{j=t_a+1}^{n} a_j} \\ &= \frac{q^{n-1}-1}{q-1} \cdot q^{s_{a-1}-1} \end{aligned}$$

where last equality can be seen by observing that

$$\sum_{j=t_a+1}^{n} a_j = \sum_{j=1}^{a-1} (t_j - t_{j+1})j = -(a-1)t_a + \sum_{j=1}^{a-1} t_j = s_{a-1}.$$

Therefore, when evaluated at C, the left hand side becomes $\frac{q^{n-1}-1}{q-1} \cdot q^{s_{a-1}+k-1}$. We now show that the right hand side has the same evaluation. Observe that our evaluation formula yields the following:

$$X_{J_{b,1}}(C) = \frac{q^n-1}{q-1} \cdot q^{k-1},$$

$$X_{J_{b,m}}(C) = \binom{t_m}{1}_q \cdot q^{-2m+1+k+s_{m-1}} \text{ for } 2 \leq m < a.$$

Note that $qX_{J_{b,1}}(C) = \frac{q^{k+n}-q^k}{q-1}$. Furthermore, we can simplify the following sum:

$$\begin{aligned}
\sum_{m=2}^{a-1} q^{2m-2}(q-1)X_{J_{b,m}}(C) &= \sum_{m=2}^{a-1} q^{2m-2}(q-1)\binom{t_m}{1}_q \cdot q^{-2m+1+k+s_{m-1}} \\
&= \sum_{m=2}^{a-1} (q^{t_m}-1)q^{s_{m-1}+k-1} \\
&= \sum_{m=2}^{a-1} (q^{s_m+k-1} - q^{s_{m-1}+k-1}) = q^{s_{a-1}+k-1} - q^{n+k-1}.
\end{aligned}$$

Now, observe that $X_{J_{b,a}}(C) = 0$ since $t_a = 0$. Now, let's move on next second set of terms with 3 blocks. We have,

$$q^2(q^2-1)X_{J_{b,1,1}}(C) = q^2(q^2-1)\binom{t_1}{2}_q \cdot q^{-2+k} = \frac{(q^{n-1}-1)}{q-1}(q^{k+n}-q^k).$$

We also observe that for $2 \le m < a$

$$\begin{aligned}
q^{2m+1}(q-1)X_{J_{b,m,1}}(C) &= q^{2m+1}(q-1)\binom{t_m}{1}_q\binom{t_1-1}{1}_q \cdot q^{-3+k+(m-1)(t_m-2)+\sum_{j=t_m+1}^{n} a_j} \\
&= q^{2m+1}(q-1)\frac{q^{t_m}-1}{q-1}\frac{q^{n-1}-1}{q-1} \cdot q^{k-2m-1+s_{m-1}} \\
&= \frac{q^{n-1}-1}{q-1}(q^{k+s_m} - q^{k+s_{m-1}}).
\end{aligned}$$

This implies $\sum_{m=2}^{a-1} q^{2m+1}(q-1)X_{J_{b,m,1}}(C) = \frac{q^{n-1}-1}{q-1}(q^{k+s_{a-1}} - q^{k+n})$ after simplification. Finally, since $t_a = 0$, $X_{J_{b,a,1}}(C) = 0$. We can now add up all of the terms on the RHS:

$$\begin{aligned}
&\frac{q^{n+k}-q^k}{q-1} + q^{s_{a-1}+k-1} - q^{n+k-1} + \frac{q^{n-1}-1}{q-1}(q^{n+k} - q^k + q^{s_{a-1}+k} - q^{n+k}) \\
&= \frac{q^n-1}{q-1} \cdot q^{k+s_{a-1}-1}
\end{aligned}$$

which is equal to the left hand side of (7), so we are done. $\square$

Remark 5.4.1. As observed above, the coefficients in the expansion of $X_{J_b} \cdot X_{J_{a,1}}$ where $b > a > 1$ are independent of b.

Remark 5.4.2. In the multiplication of the statistics for a matrix with one block and a matrix with two blocks, all of the resulting terms were statistics of matrices with at most 3 blocks.

We compute and prove other similar preliminary results for multi-block statistic expansions using the algorithm in Section 2. See a table in Appendix A. Given these patterns, we posit the following conjectures:

Conjecture 5.5. *Let m, n be fixed positive integers.*

$$X_{J_n} \cdot X_{J_{m,m}} = \begin{cases} q^m X_{J_{n,m}} + \sum_{k=1}^{m-1} q^{3k+m-1}(q-1)X_{J_{n,m,k}} + q^{4m}X_{J_{n,m,m}} & \text{if } n > m \\ q^{m-1}\binom{2}{1}_q X_{J_{n,m}} + \sum_{k=1}^{m-1} q^{3k+m-2}(q^2-1)X_{J_{n,m,k}} + q^{4m-2}\binom{3}{1}_q X_{J_{n,m,m}} & \text{if } n = m \\ q^{n-1}\binom{2}{1}_q X_{J_{m,m}} + \sum_{k=1}^{n-1} q^{3k+n-2}(q^2-1)X_{J_{m,m,k}} + q^{4n}X_{J_{m,m,n}} & \text{if } n < m. \end{cases}$$

Conjecture 5.6. *Let $n > m > k$ be positive integers. Then,*

$$X_{J_n} \cdot X_{J_{m,k}} = q^k X_{J_{n,k}} + \sum_{i=1}^{m-k-1} \left(q^{k+2i-1}(q-1)X_{J_{n,k+i}}\right) + q^{2m-k}X_{J_{n,m}}$$
$$+ \sum_{l=1}^{k-1} \left(q^{k+3l-1}(q-1)X_{J_{n,k,l}} + \sum_{j=1}^{m-k-1} \left(q^{k+3k+2j-2}(q-1)^2 X_{J_{n,k+j,l}}\right) + q^{2m+3l-k-1}(q-1)X_{J_{n,m,l}} \right)$$
$$+ q^{4k-2}(q^2-1)X_{J_{n,k,k}} + \sum_{p=1}^{m-k-1} \left(q^{4k+2p-1}(q-1)X_{J_{n,k+p,k}}\right) + q^{2m+2k}X_{J_{n,m,k}}.$$

Conjecture 5.7. *Consider the expansion of $X_A \cdot X_B$ where A and B are two unipotent Jordan matrices. If A and B have a and b blocks in their Jordan normal forms, respectively, then the only non-trivial contributions in the expansion of $X_A \cdot X_B$ are from matrices C with at most $a+b$ Jordan blocks.*

5.4 Identity Matrices

In this section, we consider our statistics under identity matrices and solve for the expansion coefficients for their products.

Notation. Let I_n denote an $n \times n$ identity matrix.

Lemma 5.8. $X_{I_m}(I_n) = \binom{n}{m}_q$.

Proof. Every m-dimensional subspace satisfies the conditions, and therefore, we are simply counting the number of m-dimensional subspaces in a n-dimensional space. By definition, this is precisely $\binom{n}{m}_q$. □

As identity matrices result in q-binomial coefficients, solving for the product expansion seeks to answer a very classical question: how do the product of two q-binomial coefficients expand in terms of a linear combination of q-binomial coefficients.

Theorem 5.9. *For $n, m \geq 1$, we have the following expansions:*

$$X_{I_m} \cdot X_{I_n} = \sum_{k=0}^{\min(m,n)} \binom{m+n-k}{k}_q \binom{m+n-2k}{m-k}_q \cdot q^{(m-k)(n-k)} X_{I_{m+n-k}}.$$

Proof. The equality is a consequence of the following counting argument. Observe that the left-hand side is counting the number of ways to pick two subspaces V and W of dimension m and n respectively.

The right-hand side counts the same thing, but by first fixing the spaces $V \cap W$ and $V + W$ then iterating through all pairs of V and W that could result in those. In this count, we start by grouping choices for subspaces V and W by the dimension $k := \dim V \cap W$. Ranging over all pairs of V and W, the dimension k takes integer values $0 \leq k \leq \min(m, n)$. Consider all pairs (V, W) with a given sum U and intersection $U' \leq U$. The possible dimensions of these are $m+n-k$ and k, respectively, by the Dimension Sum Theorem. Therefore, there are X_{m+n-k} choices for U and for each such choice there are $\binom{m+n-k}{k}_q$ choices for U'

Now for fixed $U' \leq U$ it remains to count the number of intermediate pairs $U' \leq V, W \leq U$ that together span U and intersect at U'. First, m-dimensional spaces $U' \leq V \leq U$ are in bijection with $m-k$-dimensional subspaces $V' \leq U/U'$, so there are $\binom{n+m-2k}{m-k}$ choices for such V'. For every one of these choices, spaces W such that $V \cap W = U'$ and $V + W = U$ are in bijection with direct complements W' s.t. $V' \oplus W' = U/U'$. To count complements, fix W'_0, then the set of all complements is in bijection with $\hom(W'_0, V')$, which has cardinality $q^{(m-k)(n-k)}$.

Putting all of this together, we get our right-hand side:

$$\sum_{k=0}^{\min(m,n)} \binom{m+n-k}{k}_q \binom{m+n-2k}{m-k}_q \cdot q^{(m-k)(n-k)} X_{I_{m+n-k}}.$$

$\square$

Remark 5.9.1. In the limit where $q = 1$ subspaces reduce to subsets. Our formula specializes to the following well-known formula for expressing the product of binomials as a linear combination:

$$\binom{r}{n}\binom{r}{m} = \sum_{k=0}^{\min(m,n)} \binom{r}{m+n-k}\binom{m+n-k}{k, n-k, m-k} \tag{8}$$

This formula can be proven by a counting argument analogous to the one in Theorem 5.9, though this special case if far simpler.

Now, we show that Theorem 5.9 reduces to Eq. (8). Plugging in $q = 1$, the q-binomial coefficients become classical binomial coefficients. Therefore, we have

$$\binom{r}{n}\binom{r}{m} = \sum_{k=0}^{\min(m,n)} \binom{r}{m+n-k}\binom{m+n-k}{k}\binom{m+n-2k}{m-k} \tag{9}$$

$$= \sum_{k=0}^{\min(m,n)} \binom{r}{m+n-k}\binom{m+n-k}{k, n-k, m-k}. \tag{10}$$

Thus recovering Eq. (8).

6 Application: Correlations

Our product expansion coefficients allow calculations of expectations of products of q-character polynomials, which could be used in calculating joint moments such as correlation. We first discuss a method for determining the expected value of each statistic X_B.

Suppose B is an invertible $m \times m$ matrix. It is shown in [7] that the expected value of X_B over all $n \times n$ invertible matrices A is independent of n once $n \geq m$. We may thus calculate this expectation by considering the simple case when $n = m$. In this case, $X_B(A) = 1$ whenever $A \sim B$ and $X_B(A) = 0$ otherwise. Consider $\mathrm{Gl}_n(\mathbb{F}_q)$ and the group action of conjugation. The size of the conjugacy class that contains B is the orbit of B or $\mathrm{Orb}(B)$.

Therefore, we have the following proposition.

Proposition 6.1. *For every $n \geq \dim(B)$, the random variable X_B on the uniform probability space $\mathrm{Gl}_n(F_q)$ has expectation*

$$\mathbb{E}[X_B] = \frac{\mathrm{Orb}(B)}{|\mathrm{Gl}_n(\mathbb{F}_q)|} = \frac{1}{|\mathrm{Stab}(B)|}.$$

The second equality is due to the Orbit-Stabilizer, so it remains to determine the stabilizer of B. Here, we are considering the group $\mathrm{Gl}_n(\mathbb{F}_q)$ and the group action of conjugation. The stabilizer of B is then $\mathrm{Stab}(B) = \{P \in \mathrm{Gl}_n(\mathbb{F}_q) \mid B = P^{-1}BP\}$. For this group action, the stabilizer is equivalent to the centralizer, defined to be $C_G(B) = \{P \in \mathrm{Gl}_n(\mathbb{F}_q) \mid PB = BP\}$. Solutions to this equation can be determined explicitly. For example, one can verify the following specific results.

Proposition 6.2. *For matrices with a single Jordan block, the order of the centralizer is* $|C_G(J_b)| = q^{b-1}(q-1)$. *For matrices with* 2 *Jordan blocks,* $|C_g(J_{b,a})| = q^{3a+b-2}(q-1)^2$ *where* $b \neq a$. *Furthermore,* $|C_g(J_{b,b})| = q^{4b-4}(q^2-1)(q^2-q)$.

Therefore, $\mathbb{E}[J_b] = \frac{1}{q^{b-1}(q-1)}$, $\mathbb{E}[J_{b,a}] = \frac{1}{q^{3a+b-2}(q-1)^2}$ where $b \neq a$, and $\mathbb{E}[J_{b,b}] = \frac{1}{q^{4b-4}(q^2-1)(q^2-q)}$.

As we have a method to determine expectations of statistics, it is natural to consider joint moments of these statistics such as correlation.

The correlation coefficient ρ_{XY} of the random variables X and Y is the measure of association of X and Y. It is between -1 and 1. When $|\rho|$ is larger, X and Y have a stronger relationship. Mathematically, we have the following equation.

Definition 3. The **correlation** coefficient is

$$\rho_{XY} = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sqrt{(\mathbb{E}[X^2] - \mathbb{E}[X]^2)(\mathbb{E}[Y^2] - \mathbb{E}[Y]^2)}}.$$

To calculate expectations of products such as $\mathbb{E}[XY]$ or $\mathbb{E}[X^2]$, we can first apply our product expansions to write the product as a linear combination. Then, we can determine the expected value of each term individually and combine the results with linearity of expectation. Therefore, we can fully describe the correlation between statistics of single Jordan blocks with this method.

Proof of Theorem 1.6. By definition

$$\rho = \frac{\mathbb{E}[X_{J_a}X_{J_b}] - \mathbb{E}[X_{J_a}]\mathbb{E}[X_{J_b}]}{\sqrt{(\mathbb{E}[X_{J_a}^2] - \mathbb{E}[X_{J_a}]^2)(\mathbb{E}[X_{J_b}^2] - \mathbb{E}[X_{J_b}]^2)}}.$$

First, consider $b > a$. It suffices to determine $\mathbb{E}[X_{J_a}X_{J_b}]$ and $\mathbb{E}[X_{J_a}^2]$. For the former, note from Theorem 5.1,

$$\begin{aligned}
\mathbb{E}[X_{J_a}X_{J_b}] &= \mathbb{E}[X_{J_b} + q(q-1)X_{J_{b,1}} + \cdots + q^{2a-3}(q-1)X_{J_{b,a-1}} + q^{2a}X_{J_{b,a}}] \\
&= \mathbb{E}[X_{J_b}] + \sum_{i=1}^{a-1} q^{2i-1}(q-1)\mathbb{E}[X_{J_{b,i}}] + q^{2a}\mathbb{E}[X_{J_{b,a}}] \\
&= \frac{1}{q^{b-1}(q-1)} + \sum_{i=1}^{a-1} \frac{q^{2i-1}(q-1)}{q^{3i+b-2}(q-1)^2} + \frac{q^{2a}}{q^{3a+b-2}(q-1)^2} \\
&= \sum_{i=0}^{a-1} \frac{1}{q^{b+i-1}(q-1)} + \frac{1}{q^{a+b-2}(q-1)^2} \\
&= \frac{q^a - 1}{q^{a+b-2}(q-1)^2} + \mathbb{E}[X_{J_a}]\mathbb{E}[X_{J_b}].
\end{aligned}$$

Therefore, $\mathbb{E}[X_{J_a}X_{J_b}] - \mathbb{E}[X_{J_a}]\mathbb{E}[X_{J_b}] = \frac{q^a-1}{q^{a+b-2}(q-1)^2}$. To find $\mathbb{E}[X_{J_a}^2]$, we use a similar method.

$$\begin{aligned}
\mathbb{E}[X_{J_a}^2] &= \mathbb{E}[X_{J_a} + q(q-1)X_{J_{a,1}} + \cdots + q^{2a-3}(q-1)X_{J_{a,a-1}} + q^{2a-1}(q+1)X_{J_{a,a}}] \\
&= \mathbb{E}[X_{J_a}] + \sum_{i=1}^{a-1} q^{2i-1}(q-1)\mathbb{E}[X_{J_{a,i}}] + q^{2a-1}(q+1)\mathbb{E}[X_{J_{a,a}}] \\
&= \frac{1}{q^{a-1}(q-1)} + \sum_{i=1}^{a-1} \frac{q^{2i-1}(q-1)}{q^{3i+a-2}(q-1)^2} + \frac{q^{2a-1}(q+1)}{q^{4a-4}(q^2-q)(q^2-1)} \\
&= \frac{q^a - 1}{q^{2a-2}(q-1)^2} + \mathbb{E}[X_{J_a}]^2.
\end{aligned}$$

Therefore, $\mathbb{E}[X_{J_a}^2] - \mathbb{E}[X_{J_a}]^2 = \frac{q^a-1}{q^{2a-2}(q-1)^2}$ and similarly for b. Thus,

$$\rho = \frac{\frac{q^a-1}{q^{a+b-2}(q-1)^2}}{\sqrt{\frac{q^a-1}{q^{2a-2}(q-1)^2} \cdot \frac{q^b-1}{q^{2b-2}(q-1)^2}}} = \sqrt{\frac{q^a-1}{q^b-1}}.$$

The case when $b = a$ is identical and reduces to a correlation of 1, as desired. $\square$

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Appendix A Calculations

Term 1	Term 2	Product Expansion
X_{J_n}	$X_{J_{2,2}}$	$q^2X_{J_{n,2}} + (q-1)q^4X_{J_{n,2,1}} + q^8X_{J_{n,2,2}}$
X_{J_n}	$X_{J_{3,2}}$	$q^2X_{J_{n,2}} + q^4X_{J_{n,3}} + (q-1)q^4X_{J_{n,2,1}} + (q^2-1)q^6X_{J_{n,2,2}}$ $+(q-1)q^6X_{J_{n,3,1}} + q^{10}X_{J_{n,3,2}}$
X_{J_n}	$X_{J_{4,2}}$	$q^2X_{J_{n,2}} + (q-1)q^3X_{J_{n,3}} + q^6X_{J_{n,4}} + q^4(q-1)X_{J_{n,2,1}}$ $+(q-1)^2q^5X_{J_{n,3,1}} + (q^2-1)q^6X_{J_{n,2,2}} + q^9(q-1)X_{J_{n,3,2}}$ $+q^8(q-1)X_{J_{n,4,1}} + q^{12}X_{J_{n,4,2}}$
X_{J_n}	$X_{J_{5,2}}$	$q^2X_{J_{n,2}} + 10q^4X_{J_{n,2,1}} + (q-1)q^3X_{J_{n,3}} + (q^2-1)q^6X_{J_{n,2,2}}$ $+(q-1)^2q^5X_{J_{n,3,1}} + (q-1)q^5X_{J_{n,4}} + (q-1)q^9X_{J_{n,3,2}}$ $+(q-1)^2q^7X_{J_{n,4,1}} + q^8X_{J_{n,5}} + (q-1)q^{11}X_{J_{n,4,2}}$ $+(q-1)q^{10}X_{J_{n,5,1}} + q^{14}X_{J_{n,5,2}}$
X_{J_n}	$X_{J_{6,2}}$	$q^2X_{J_{n,2}} + (q-1)q^4X_{J_{n,2,1}} + (q-1)q^3X_{J_{n,3}} + (q^2-1)q^6X_{J_{n,2,2}}$ $+(q-1)^2q^5X_{J_{n,3,1}} + (q-1)q^5X_{J_{n,4}} + (q-1)q^9X_{J_{n,3,2}}$ $+(q-1)^2q^7X_{J_{n,4,1}} + (q-1)q^7X_{J_{n,5}} + (q-1)q^{11}X_{J_{n,4,2}}$ $+(q-1)^2q^9X_{J_{n,5,1}} + q^{10}X_{J_{n,6}} + (q-1)q^{13}X_{J_{n,5,2}}$ $+(q-1)q^{12}X_{J_{n,6,1}} + q^{16}X_{J_{n,6,2}}$
X_{J_n}	$X_{J_{3,3}}$	$q^3X_{J_{n,3}} + (q-1)q^5X_{J_{n,3,1}} + (q-1)q^8X_{J_{n,3,2}} + q^{12}X_{J_{n,3,3}}$
X_{J_n}	$X_{J_{4,3}}$	$q^3X_{J_{n,3}} + q^5X_{J_{n,4}} + (q-1)q^5X_{J_{n,3,1}} + (q-1)q^8X_{J_{n,3,2}}$ $+(q-1)q^7X_{J_{n,4,1}} + (q^2-1)q^{10}X_{J_{n,3,3}} + (q-1)q^{10}X_{J_{n,4,2}} + q^{14}X_{J_{n,4,3}}$
X_{J_n}	$X_{J_{5,3}}$	$q^3X_{J_{n,3}} + (q-1)q^5X_{J_{n,3,1}} + (q-1)q^4X_{J_{n,4}} + (q-1)q^8X_{J_{n,3,2}}$ $+(q-1)^2q^6X_{J_{n,4,1}} + q^7X_{J_{n,5}} + (q^2-1)q^{10}X_{J_{n,3,3}} + (q-1)^2q^9X_{J_{n,4,2}}$ $+(q-1)q^9X_{J_{n,5,1}} + (q-1)q^{13}X_{J_{n,4,3}} + (q-1)q^{12}X_{J_{n,5,2}} + q^{16}X_{J_{n,5,3}}$
X_{J_n}	$X_{J_{4,4}}$	$q^4X_{J_{n,4}} + (q-1)q^6X_{J_{n,4,1}} + (q-1)q^9X_{J_{n,4,2}}$ $+(q-1)q^{12}X_{J_{n,4,3}} + q^{16}X_{J_{n,4,4}}$
X_{J_n}	$X_{J_{2,2,2}}$	$q^4X_{J_{n,2,2}} + (q-1)q^7X_{J_{n,2,2,1}} + q^{12}X_{J_{n,2,2,2}}$
X_{J_n}	$X_{J_{3,3,2}}$	$q^5X_{J_{n,3,2}} + (q-1)q^8X_{J_{n,3,2,1}} + q^8X_{J_{n,3,3}}$ $+(q-1)q^{11}X_{J_{n,3,3,1}} + (q^2-1)q^{11}X_{J_{n,3,2,2}} + q^{16}X_{J_{n,3,3,2}}$

Table 1: Table of Product Expansions

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