On the Hausdorff Dimension of the Visible Koch Curve

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Abstract

In geometry, a point in a set is visible from another point if the line segment connecting two points does not contain other points in the set. We show that the Hausdorff dimension is 1 for the portion of the Koch curve that is visible from points at infinity and points in certain defined regions of the plane.

1 Introduction

The Koch curve was first described by Helge von Koch in 1904 as an example of a continuous but nowhere differentiable curve. It is a bounded fractal on the plane with infinite length. The Hausdorff dimension was introduced in 1918 by Felix Hausdorff as a generalization of the usual sense of integral dimension.

In this paper, we concentrate on the Hausdorff dimension of the parts of the Koch curve visible from a given point. This project was suggested by Prof. Larry Guth. To our knowledge, there has not been past literature on the study the visibility of a fractal. We show that the Hausdorff dimension is 1 from points of visibility at infinity and from a set of points in certain regions of $\mathbb{R}^2$. We also show that from any point in $\mathbb{R}^2$ the Hausdorff dimension is bounded below by 1.

In Section 2, we discuss Hausdorff dimension and its properties. In Section 3, we define Koch curve and the notion of point visibility used in this paper. Finally, we calculate the Hausdorff dimension of the visible Koch curve from points at infinity and certain points in $\mathbb{R}^2$ in Section 4 and Section 5 respectively.

2 Hausdorff Dimension

In this section, we give the definitions for basic concepts involved in defining the Hausdorff dimension. Our exposition in the subsections follows [1].

2.1 Hausdorff Measure

Consider a subset U of $\mathbb{R}^n$. The *diameter* of U is defined to be $|U| = \sup\{|x-y| : x, y \in U\}$, i.e., the least upper bound for the distances between any two points in U. A δ-*cover* for a set $F \subset \mathbb{R}^n$ is a countable collection of sets with diameter at most δ that covers F, i.e.,

$$F \subset \bigcup_{i=0}^{\infty} U_i$$

where $|U| \leq \delta$. We define the Hausdorff measure of a set using covers with increasingly small radius approaching 0.

Definition 2.1. The *s-dimensional Hausdorff measure* of a set $F \subset \mathbb{R}^n$ is defined to be

$$\mathcal{H}^s = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F),$$

where $\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$.

The Hausdorff measure generalizes the notion of length, area, and volume. More formally, the n-dimensional Hausdorff measure for a Borel set $B \subset \mathbb{R}^n$ is a constant multiple of n-dimensional Lebesgue measure λ, that is,

$$\mathcal{H}^n(B) = V_n \lambda(B),$$

where V_n is the volume of the n-ball with diameter 1 [1].

To see how the Hausdorff measure induces the same kind of scaling behavior as familiar notions of length, area, and volume, we note how the Hausdorff measure of a set F changes when it undergoes a similarity transformation, that is, a mapping S such that

$$|S(x) - S(y)| = c|x - y|$$

for some scaling factor $c > 0$ for all $x, y \in F$.

Proposition 2.1. *Let S be a similarity transformation of scaling factor c, then*

$$\mathcal{H}^s(S(F)) = c^s \mathcal{H}^s(F).$$

Next, we exhibit some useful basic properties of Hausdorff measures. To do that, we first define several notions related to relative distance for transformations of a set in a metric space.

Definition 2.2. A function f satisfies the *Hölder condition* if

$$|f(x) - f(y)| \leq c|x - y|^\alpha$$

for non-negative real constants $c, \alpha \geq 0$.

Definition 2.3. A function f is *Lipschitz* if it satisfies the Hölder condition with exponent $\alpha = 1$, i.e.,

$$|f(x) - f(y)| \leq c|x - y|$$

where $c \geq 0$.

By directly applying the definition of Hausdorff measures, we arrive at the following fact on the transformation property of Hausdorff measures:

Proposition 2.2. *Let $F \in \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ be a mapping that satisfies a Hölder condition*

$$|f(x) - f(y)| \leq c|x - y|^\alpha$$

for constants $c, \alpha \geq 0$. Then for each s,

$$\mathcal{H}^{\frac{s}{\alpha}}(f(F)) \leq c^{\frac{s}{a}} \mathcal{H}^s(F).$$

2.2 Hausdorff Dimension

Now equipped with the notion of Hausdorff measures, we can finally define the Hausdorff dimension of a set. We first note that for a set F, the t-dimensional Hausdorff measure vanishes to 0 when the s-dimensional Hausdorff measure is finite for some $s < t$. Indeed, let $\{U_i\}$ be a δ-cover of F, we have

$$\sum_i |U_i|^t = \sum_i |U_i|^{t-s}|U_i|^s \leq \delta^{t-s} \sum_i |U_i|^s.$$

Since taking the infimum preserves the inequality, $\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_d(F)$. Letting $\delta \to 0$, we have $\mathcal{H}^t(F) = 0$ when $t > s$ and $\mathcal{H}^s(F) < \infty$, establishing our claim. We see that there exists a critical value for the dimension of the Hausdorff measure of a set at which the measure changes from ∞ to 0. The Hausdorff dimension is defined to be this critical value. The formal definition follows.

Definition 2.4. The *Hausdorff dimension* $\dim_{\mathrm{H}} F$ of a set $F \in \mathbb{R}^n$ is defined to be

$$\dim_{\mathrm{H}} F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

Using Proposition 2.2, we can quickly establish the following property of the Hausdorff dimension which will be helpful later in calculating the Hausdorff dimension of certain fractal sets.

Proposition 2.3. *Let $F \in \mathbb{R}^n$ and suppose that $f : F \to \mathbb{R}^m$ satisfies a Hölder condition*

$$|f(x) - f(y)| \leq c|x - y|^\alpha,$$

then $\dim_{\mathrm{H}} f(F) \leq \frac{1}{\alpha}\dim_{\mathrm{H}} F$.

Corollary 2.3.1. *If $f : F \to \mathbb{R}^m$ is Lipschitz, then* $\dim_{\mathrm{H}} f(F) \leq \dim_{\mathrm{H}} F$.

Corollary 2.3.2. *If $f : F \to \mathbb{R}^m$ is a bi-Lipschitz transformation such that*

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y|,$$

where $0 < c_1 \leq c_2 < \infty$, then $\dim_{\mathrm{H}} f(F) = \dim_{\mathrm{H}} F$.

2.3 Hausdorff Dimension of Self-Similar Fractals

Most commonly seen and studied fractals involve some degree of self-similarity, and the calculations of Hausdorff dimension for these fractals can be greatly simplified due to the following result for self-similar fractals.

First, we make rigorous the concept of self-similarity for fractals using contraction maps and iterated function system.

Definition 2.5. For a closed set D, a mapping $S : D \to D$ is a *contraction* on D if there exists a constant $0 < c < 1$ such that

$$|S(x) - S(y)| \leq c|x - y|$$

for all $x, y \in D$. If equality holds, then S is called a *contraction similarity.*

Definition 2.6. An *iterated function system* (IFS) is a finite family of contraction maps $\{S_1, S_2, \ldots, S_m\}$.

If we repeatedly apply an iterated function system to a set F, then the resulting set approaches a "limit" set, or the attractor for the IFS. More rigorously put, the *attractor* of an IFS is a compact subset F of D such that

$$F = \bigcup_{i=1}^{m} S_i(F).$$

It can be shown that the attractor of an IFS is unique (see [1]), so a fractal can be defined uniquely with an IFS.

Given the concept of contraction similarities, we define a self-similar set as follows.

Definition 2.7. A self-similar set is the attractor of the iterated function system $\{S_1, S_2, \ldots, S_m\}$ such that each S_i is a similarity, i.e., $|S_i(x) - S_i(y)| = c|x - y|$ for some $0 < c < 1$.

Essentially, a self-similar set is a union of smaller similar copies of itself, and thus it is unchanged when the iterated function system is applied.

The Hausdorff dimension of a self-similar set is easy to calculate due to the following theorem provided that the similarity maps S_i in the IFS satisfies the *open set condition*, which requires the existence of a nonempty bounded open set V such that

$$\bigcup_{i=1}^{m} S_i(V) \in V.$$

Proposition 2.4. *If F is a self-similar set, i.e., the attractor of an IFS $\{S_1, S_2, \ldots, S_m\}$ where S_i are contraction similarities satisfying the open set condition with scaling factor $0 < c_i < 1$ for all $1 \leq i \leq m$, then $\dim_{\mathrm{H}} F = s$, where s is given by*

$$\sum_{i=1}^{m} c_i^s = 1.$$

Moreover, for this value of s, we have $0 < \mathcal{H}^s(F) < \infty$.

While a full proof is contained in [1], we include here a short heuristic argument [1] presented to motivate the result.

Since the set of similarities S_i satisfy the open set condition, the similar copies of the set F do not "overlap too much," which is to say that the union $\bigcup_{i=1}^{m} S_i(F)$ is "nearly disjoint." Then we have

$$\mathcal{H}^s(F) = \sum_{i=1}^{m} \mathcal{H}^s(S_i(F)) = \sum_{i=1}^{m} c_i^s \mathcal{H}^s(F)$$

where the last equality is given by the scaling property of Hausdorff measures, Proposition 2.1. Assuming that $0 < \mathcal{H}^s(F) < \infty$ at this critical value of $\dim_{\mathrm{H}} = s$, we have $\sum_{i=1}^{m} c_i^s = 1$ as in Proposition 2.4.

3 Koch Curve and Visibility

First, we define the Koch curve K. The Koch curve can be constructed by iteratively replacing the middle third of each line segment by an equilateral triangle of the same length with the base removed, as shown in Figure 3.1.

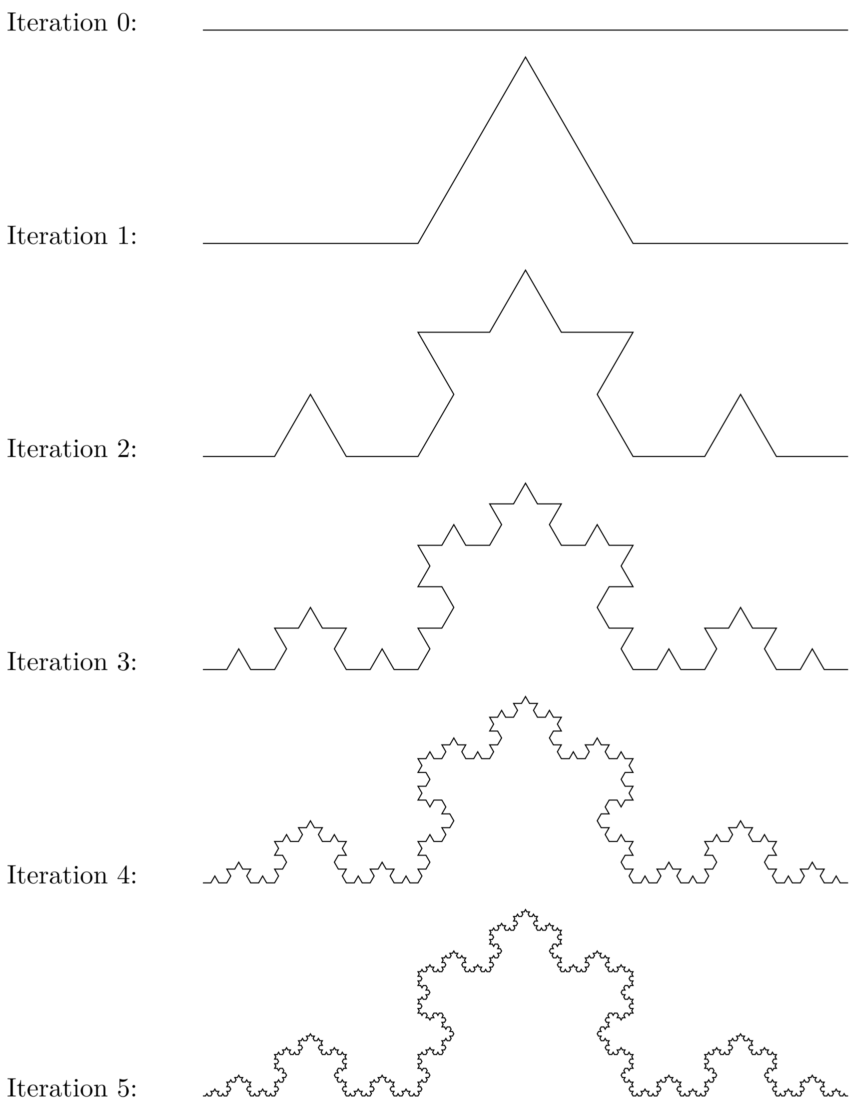


Figure 3.1: The iterative construction of the Koch curve.

Formally as a self-similar fractal, the Koch curve can be defined via an iterated function system.

Definition 3.1. The *Koch curve* $K \subset \mathbb{C}$ is defined by the following IFS $\{S_1, S_2, S_3, S_4\}$. Let $\omega = e^{\frac{\pi i}{3}}$ be the sixth root of unity,

$$S_1(z) = \frac{z}{3}; \quad S_2(z) = \frac{\omega z + 1}{3}; \quad S_3(z) = \frac{\omega^2 x + 2}{3}; \quad S_4(z) = \frac{z+2}{3}.$$

In the following discussion, the Koch curve is positioned in $\mathbb{R}^2$ such that the base segment (Iteration 0) is placed on the interval $[0, 1]$ on the x-axis, so the curve is centered along the line $x = \frac{1}{2}$.

Since the Koch curve is self-similar, we can apply Proposition 2.4 to calculate to calculate its Hausdorff dimension. Note that the scaling factor c_i for the similarity transformations S_i is $\frac{1}{3}$ for each $1 \leq i \leq 4$. Thus, the Hausdorff dimension of the Koch curve $\dim_{\mathrm{H}}(K) = s$ satisfies

$$\sum_{i=1}^{4} c_i^s = \sum_{i=1}^{4} \frac{1}{3^s} = \frac{4}{3^s} = 1,$$

from which we obtain that $\dim_{\mathrm{H}}(K) = \log_3 4$, a classical result in fractal geometry.

Next, we define the concept of visibility for a set from a given a point.

Definition 3.2. A point P in a set S is *visible* from a point V if there are no other points in S on the line segment connecting P and V. The collection of all points in S visible from V is denoted S_V. The line connecting the point of visibility V to a point $p \in S$ is the *line of visibility* through P.

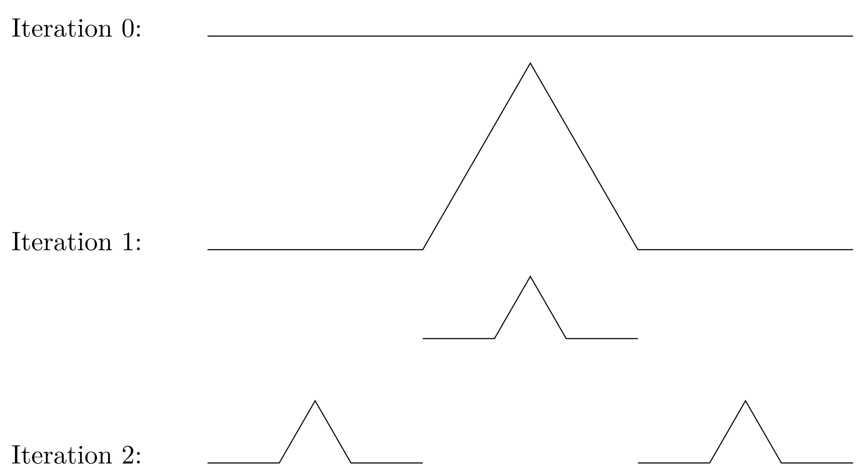
The goal is to calculate the Hausdorff dimension of the portion of the Koch curve visible from different points in the plane.

4 Visibility from points at infinity

4.1 Visibility from $(1/2, \infty)$

Figure 4.1 gives the portion of the first few iterations of the Koch curve that is visible from $(\frac{1}{2}, \infty)$, denoted as $K_{(\frac{1}{2}, \infty)}$.

The intersection of the Koch curve with the initial line segment (Iteration 0) is visible—it is the Cantor set.



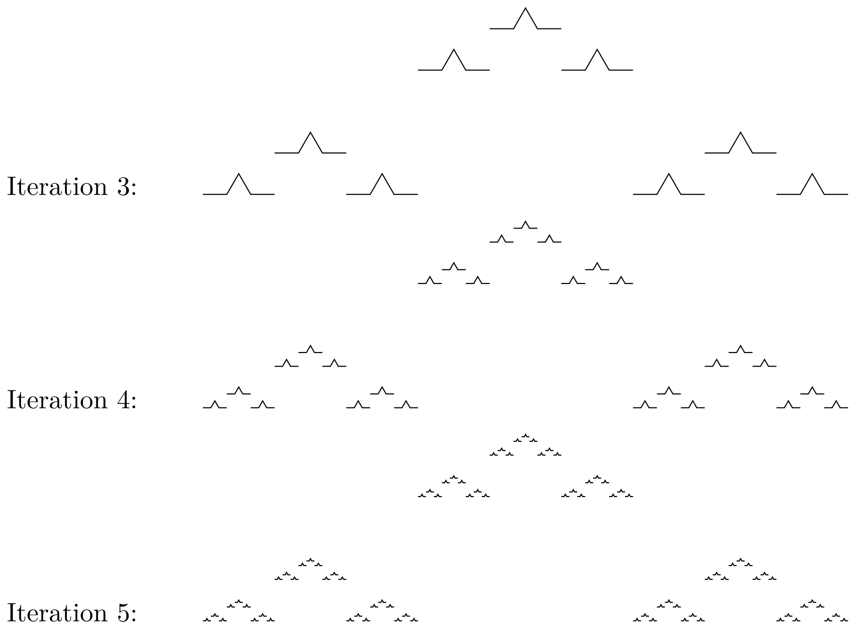
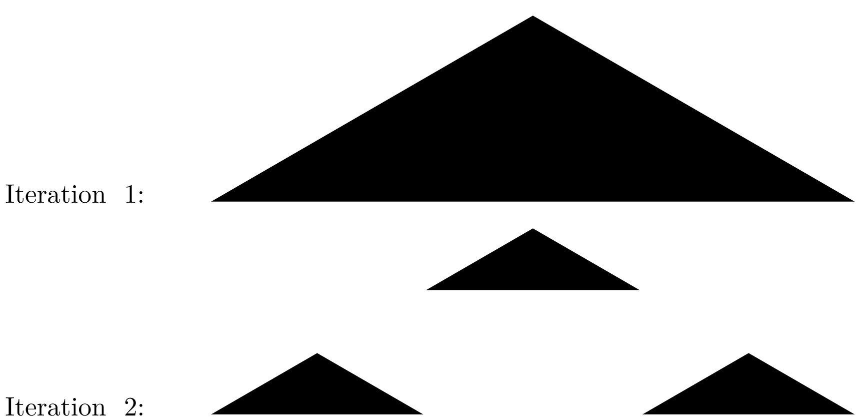


Figure 4.1: Iterations of the Koch curve visible from $(\frac{1}{2}, \infty)$

We aim to calculate the Hausdorff dimension $\dim_{\mathrm{H}} K_{(\frac{1}{2},\infty)}$ of the new fractal. Before then, we first note the similarity of $K_{(\frac{1}{2},\infty)}$ with the Sierpinski gasket from the iterations of $K_{(\frac{1}{2},\infty)}$ (Figure 4.1). We will formalize this similarity here.

Let us start with a filled triangle with vertices at $(0,0)$, $(1,0)$, and $(\frac{1}{2}, \frac{\sqrt{3}}{6})$, which is the convex hull of the Koch curve. We define a modified gasket by an iterative process. In each iteration we start with a triangle. Then we divide each side into three equal segments. We connect two division points close to one vertices of a triangle by a line. This way the original triangle is divided into three corner triangle and a hexagon. We now replace the initial triangle with three corner triangles. Equivalently, we remove the hexagon in the center. The first few iterations of the modified Sierpinski gasket are shown in Figure 4.2.



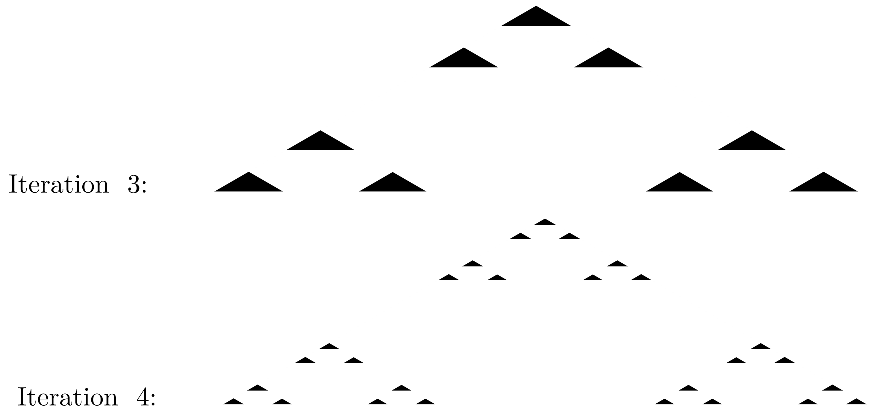


Figure 4.2: Iterations of the modified Sierpinski gasket

It's clear from the iteration process that the modified gasket can be defined by an IFS with 3 similarities with scaling factor $\frac{1}{3}$. Applying Proposition 2.1 gives that the Hausdorff dimension of the modified gasket is 1. Note that the visible portion of the Koch curve from $(\frac{1}{2}, \infty)$ is the intersection of the Koch curve with the modified gasket.

Proposition 4.1. *The Hausdorff dimension of the Koch curve visible from* $(\frac{1}{2}, \infty)$ *is 1:*

$$\dim_{\mathrm{H}} K_{(\frac{1}{2},\infty)} = 1.$$

Proof. Let K^n denotes the fractal $K_{(\frac{1}{2},\infty)}$ at iteration n. The length, or equivalently, the 1-dimensional Hausdorff measure $\mathcal{H}^1(K^1)$ of the fractal K^1 is $\frac{4}{3}$ as it consists of four congruent segments of length $\frac{1}{3}$. Observe that the length of the fractal K_i is preserved in each iteration $i > 1$ since K_{i+1} consists of three copies of K^i with scaling factor $\frac{1}{3}$, so $\mathcal{H}^1(K^{i+1}) = \mathcal{H}^1(K^i)$. Letting $i \to \infty$, we have $\mathcal{H}^1(K_{(\frac{1}{2},\infty)}) = \frac{4}{3}$. Then $0 < \mathcal{H}^1(K_{(\frac{1}{2},\infty)}) < \infty$, and $\dim_{\mathrm{H}} K_{(\frac{1}{2},\infty)} = 1$. □

4.2 Visibility from a general point at infinity

After we have discussed a specific point of visibility in section 4.1, we are ready to prove the theorem for the more general case.

Theorem 4.2. *The Hausdorff dimension of the Koch curve from any point of visibility at a direction of infinity* K_{V_∞} *is 1:*

$$\dim_{\mathrm{H}} K_{V_\infty} = 1.$$

Proof. Let l be a line perpendicular to the parallel lines of visibility from the point of visibility V to K_V that separates V from K_V, i.e., all the points in K_V lies on a different side of l than V. Consider the map $f : K_V \mapsto l$ is a orthogonal projection of K_V onto l. We claim that at each iteration of the curve K_V^n, its length is finite and bounded. Note that K_V^n contains only three sets of parallel line segments, and their projections onto l do not overlap except at end points. Let $I = f(K_V)$ be the line segment K_V^n is projected onto, and $\overrightarrow{u_l}$ and

$\overrightarrow{u_i}$ be unit vectors in the direction of l and u_i with angles of inclination $i = 0°, 60°, 120°$. then the length of K_V^n is bounded by:

$$m\mathcal{H}^1(K_V^n)I \leq \mathcal{H}^1(K_V^n) \leq M\mathcal{H}^1(K_V^n)I$$

where $m = \min_{\overrightarrow{u_l}\cdot\overrightarrow{u_i}>0}(\overrightarrow{u_l} \cdot \overrightarrow{u_i})$ and $M = \max(\overrightarrow{u_l} \cdot \overrightarrow{u_i})$. Therefore, we have

$$0 < \mathcal{H}^1(K_V^n) < M\mathcal{H}^1(K_V^n)I,$$

as claimed. □

5 Lower bound on the Hausdorff dimension for visibility points in $\mathbb{R}$

In this section, we show that the Hausdorff dimension of the Koch curve visible from a finite point is bounded below by 1.

Proposition 5.1. *The Hausdorff dimension of the Koch curve visible from a point $v \in \mathbb{R}^2$ that does not lie on the Koch curve is bounded below by 1:*

$$\dim_{\mathrm{H}} K_V \geq 1.$$

Proof. Consider the projection map $f : K_V \to \Omega$ where Ω is a circle centered at V that does not touch nor enclose parts of the Koch curve K. (This is possible because the K is closed in $\mathbb{R}^2$). Each point $P \in K_V$ is mapped to the intersection of Ω and the line PV as in Figure 5.1 and 5.2.

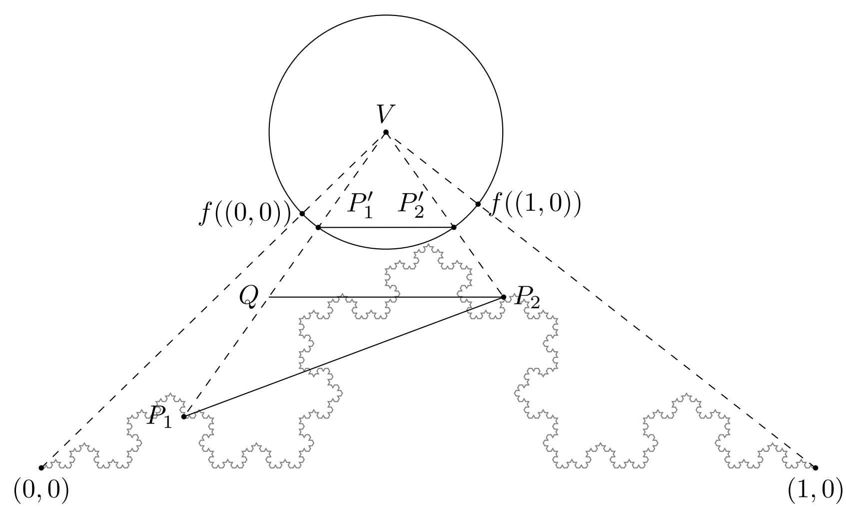


Figure 5.1

We claim that f is Lipschitz. Let P_1, P_2 be two points on K_V, and P_1', P_2' be their projections onto Ω through f. Without loss of generality, let P_2 be the closer point than P_1

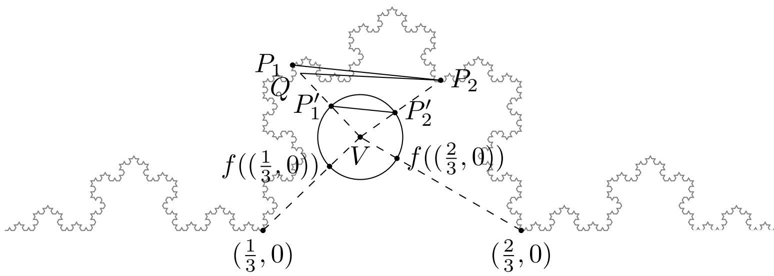


Figure 5.2

to the line $P_1'P_2'$. Let Q be the intersection of the line through P_2 parallel to $P_1'P_2'$ and the line through V and P_1. Then we have

$$|\overline{P_1'P_2'}| \leq |\overline{QP_2}| \leq |\overline{P_1P_2}|$$

where the first inequality is given by similar triangles, and the second inequality comes from the fact that triangle $VP_1'P_2'$ is isosceles and thus angle P_1QP_2 is obtuse. Since f is Lipschitz, by Corollary 2.3.1, we have $\mathcal{H}^1(K_V) \geq \mathcal{H}^1(f(K_V)) > 0$, where the last inequality comes from the fact that $\mathcal{H}^1(f(K_V))$ is the length of an arc and thus nonzero. Since $\mathcal{H}^1(K_V) > 0$, $\dim_{\mathrm{H}} K_V \geq 1$. □

6 Visibility from points in $\mathbb{R}^2$

In this section, we calculate the Hausdorff dimension of the Koch curve visible from points in the shaded region S shown in Figure 6.1.

Proposition 6.1. *The Hausdorff dimension of the Koch curve visible from a point V in region S*

$$\dim_{\mathrm{H}} K_V = 1,$$

where we define $S = R^2 - \cup_{i=1}^3 S_i$ and

$$S_1 = \left\{(x,y) \mid 0 < y < \frac{\sqrt{3}}{6}\right\},$$
$$S_2 = \left\{(x,y) \mid \sqrt{3}x - 1 < y < \sqrt{3}x\right\},$$
$$S_3 = \left\{(x,y) \mid -\sqrt{3}x < y < sqrt3x + 1\right\}.$$

Proof. We prove the theorem for the upper region S_1. The proofs for other regions are nearly identical.

Let $V \equiv (x_V, y_V)$. We show that the Hausdorff dimension of the fractal K_V is bounded above by 1 by proving that it has finite length, i.e., $\mathcal{H}^1(K_V) < \infty$, from which we can conclude by the definition of Hausdorff dimension that $\dim_{\mathrm{H}} K_V \leq 1$.

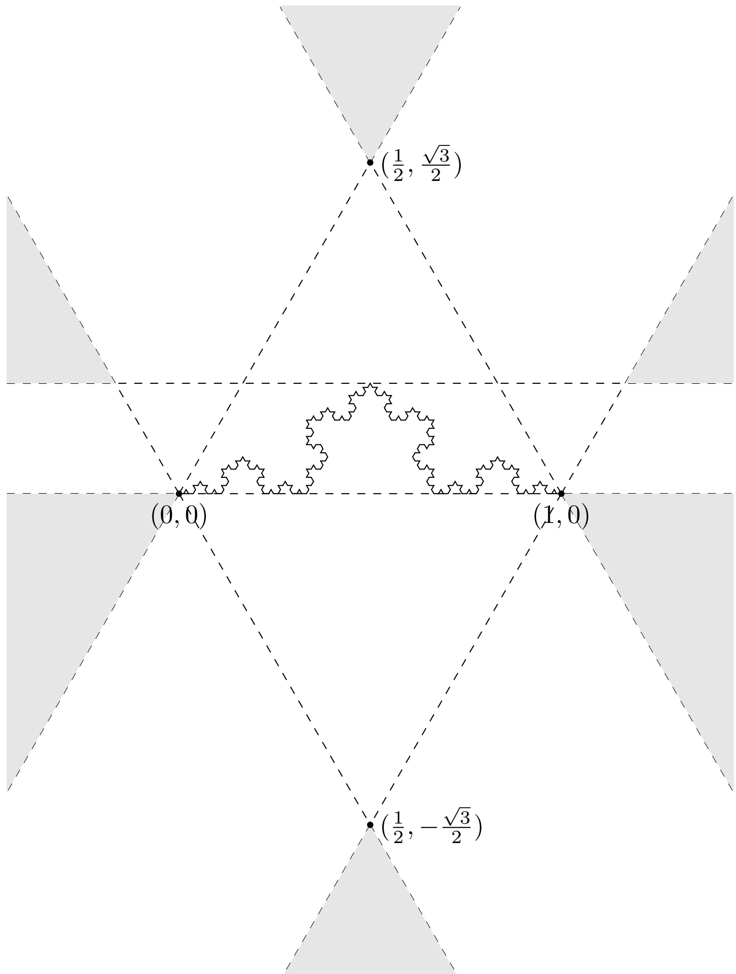


Figure 6.1: Shaded region indicates S.

Similar to the proof of Proposition 5.1, consider the projection function $f : K_V \to l$ where l is a horizontal line $y = \frac{\sqrt{3}}{6}$ that passes through the apex of K_V at $(\frac{1}{2}, \frac{\sqrt{3}}{6})$. Since V is above the equilateral triangle it forms an angle θ_0 outside of the equilateral triangle (see Figure 6.2). Our goal is to demonstrate that in each iteration n, the length of K_V^n is bounded by a constant multiple of $|f((0,0)) - f((1,0))|$, that is,

$$\mathcal{H}^1(K_V^n) \leq M|f((0,0)) - f((1,0))|$$

where M is a positive constant. We prove this by bounding the length of each line segment in the fractal with respect to the length of its image on the line l.

We consider the part of the fractal K_V^n that lies to the left of V, i.e. points on K_V with x-coordinate less than or equal to x_V. There are three cases depending on the angle of inclination of the line segment: $0°$, $60°$, and $120°$.

Case 1: The line segment is horizontal as in Figure 6.3.

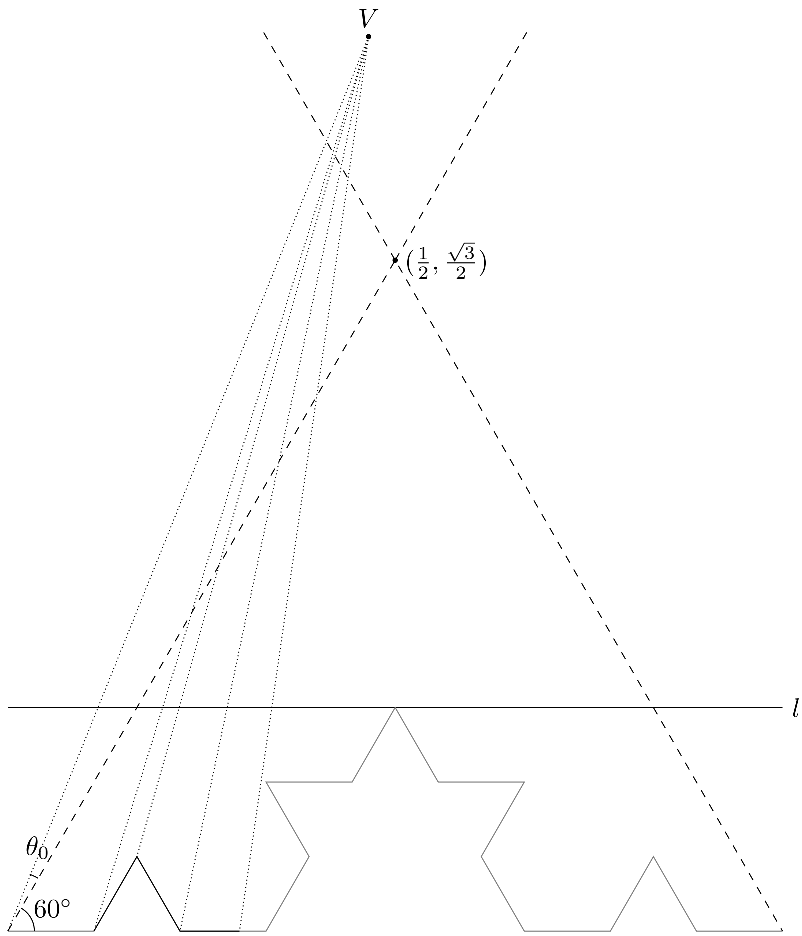


Figure 6.2

Let P_1 and P_2 be the endpoints of the line segment with coordinates (x_1, y) and (x_2, y), for $0 \leq y \leq \frac{\sqrt{3}}{6}$.

By similar triangles,

$$|P_1 - P_2| = \frac{y_V - y}{y_V - \frac{\sqrt{3}}{6}}|f(P_1) - f(P_2)| \leq \frac{y_V}{y_V - \frac{\sqrt{3}}{6}}|f(P_1) - f(P_2)|.$$

Case 2: The line segment has an angle of inclination of 60°as in Figure 6.4.

Let P_1 and P_2 be the endpoints of the line segment. We extend the line from V to P_2 until its intersects with the horizontal line at Q. Let θ denote the angle formed between the line of visibility through point P_2 and the line connecting P_1 and P_2 with 60°of inclination.

By the law of sines,

$$\frac{\sin\theta}{|P_1 - Q|} = \frac{\sin\phi}{|P_1 - P_2|}.$$

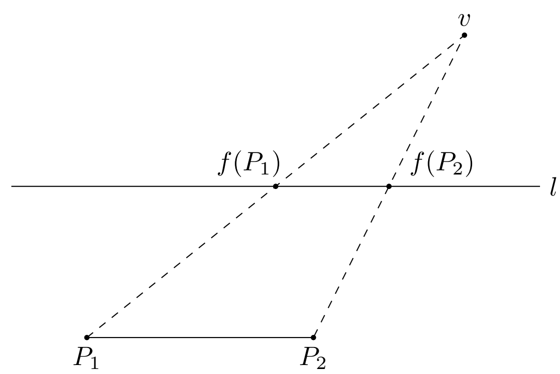


Figure 6.3

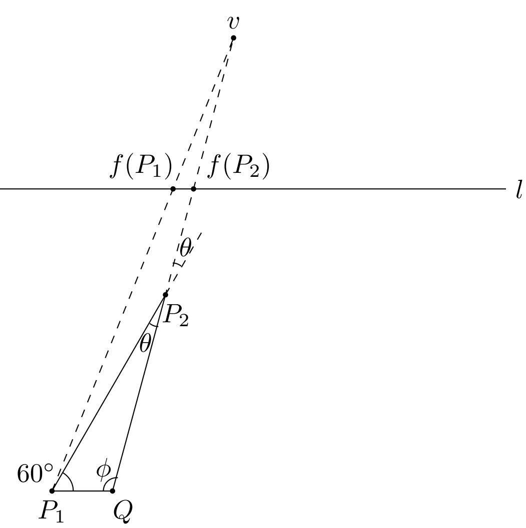


Figure 6.4

Since $\phi = 120° - \theta$, we have

$$|P_1 - P_2| = \frac{\sin(120° - \theta)}{\sin\theta}|P_1 - Q| \leq \frac{1}{2}(\sqrt{3}\cot\theta + 1)|P_1 - Q|.$$

Applying the result from Case 1,

$$|P_1 - Q| \leq \frac{a}{a - \frac{\sqrt{3}}{6}}|f(P_1) - f(P_2)|.$$

Since θ_0 is the smallest angle a line of visibility can form with lines with 60°of inclination, we must have $\theta \geq \theta_0$. Since $\cot x$ is a decreasing function for $0 \leq x \leq \frac{\pi}{2}$, we have

$$\cot\theta \leq \cot\theta_0.$$

Combining the inequalities,

$$|P_1 - P_2| \leq \frac{1}{2}(\sqrt{3}\cot\theta + 1)|P_1 - Q| \leq \frac{y_V(\sqrt{3}\cot\theta + 1)}{2(y_V - \frac{\sqrt{3}}{6})}|f(P_1) - f(P_2)|$$
$$\leq \frac{y_V(\sqrt{3}\cot\theta_0 + 1)}{2(y_V - \frac{\sqrt{3}}{6})}|f(P_1) - f(P_2)|.$$

Case 3: The line segment has an angle of inclination of 120°as in Figure 6.5.

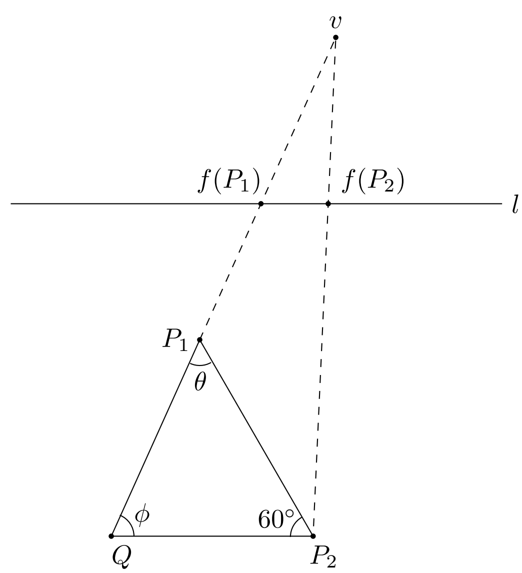


Figure 6.5

We label the points similarly as in Case 2. Now let θ denote the angle formed at the vertex P_1. Since P_1 lies in the left side of the fractal, we have $\theta > 30°$ (because $\theta = 30°$ when the line of visibility through P_1 is vertical), and thus $\cot\theta \leq \cot 30° = \sqrt{3}$. Using the same derivation as in Case 2,

$$|P_1 - P_2| \leq \frac{y_V(\sqrt{3}\cot\theta + 1)}{2(y_V - \frac{\sqrt{3}}{6})}|f(P_1) - f(P_2)| \leq \frac{2y_V}{y_V - \frac{\sqrt{3}}{6}}|f(P_1) - f(P_2)|.$$

To combine the three cases, let

$$M = \max\left(\frac{y_V}{y_V - \frac{\sqrt{3}}{6}}, \frac{y_V(\sqrt{3}\cot\theta_0 + 1)}{2(y_V - \frac{\sqrt{3}}{6})}, \frac{2y_V}{y_V - \frac{\sqrt{3}}{6}}\right),$$

which is a fixed constant since y_V and θ_0 are fixed. Then, for each line segment in the fractal K_V^n, we have

$$|P_1 - P_2| \leq M|f(P_1) - f(P_2)|$$

where P_1, P_2 are endpoints of the line segment. Since f is a bijection, the projection of distinct line segments do not overlap. Thus, adding up the lengths of each line segment in

K_V^n, we have the following bound on the total length of K_V^n,

$$\mathcal{H}^1(K_V^n) \le M|f((0,0)) - f((1,0))| = \frac{c(y_V - \frac{\sqrt{3}}{6})}{y_V}.$$

Therefore, for the limiting set $K^{(\frac{1}{2},a)}$, we have

$$\mathcal{H}^1(K_V) = \lim_{n\to\infty} \mathcal{H}^1(K_V^n) \le \frac{M(y_V - \frac{\sqrt{3}}{6})}{y_V} < \infty,$$

establishing our claim. □

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References

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