Hilbert Series of Quasiinvariant Polynomials

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Abstract

The space of quasiinvariant polynomials generalize that of symmetric polynomials: under the action of the symmetric group, the polynomials remain invariant to a certain order. We discern the structure and symmetries of quasiinvariant polynomials by way of examining the invariance of relevant polynomial spaces under certain specific group actions. Both pure and computational methods are employed in this pursuit. Felder and Veselov, when studying quasiinvariant polynomials, made a breakthrough discovery in computing their Hilbert series in fields of characteristic 0, and since then, quasiinvariant polynomials have been extensively studied due to their applications in representation theory, algebraic geometry, and mathematical physics. We investigate the Hilbert series of quasiinvariant polynomials that are divisible by a generic homogeneous polynomial. We also continue the previous work regarding their Hilbert series in fields of prime characteristic.

1 Introduction

Let us denote $s_{i,j}$ to be the operator interchanging x_i and x_j. For example, we have

$$s_{1,3}(x_1x_2^2 + x_3) = (x_3x_2^2 + x_1).$$

There is an interesting class of polynomials that remain invariant under this operator — symmetric polynomials. A symmetric polynomial σ would thus satisfy $\sigma - s_{i,j}(\sigma) = 0$ for all i, j. Equivalently, we can also describe symmetric polynomials as being invariant under the action of the symmetric group.

A class of polynomials that generalize that of symmetric polynomials have been increasingly studied by mathematicians. Labeled *m-quasiinvariant polynomials*, these polynomials, generalized by Berest and Chalykh, remain invariant under the action of complex reflection groups to a certain order [1]. Chalykh also introduced the rings of quasiinvariant polynomials in the theory of integrable systems: one of the most relevant areas in this field concerns the deep relations of quasiinvariant polynomials with the eigenfunctions of generalized Calogero-Moser systems [2].

The Calogero-Moser problem was originally studied by specialists in integrable systems, which are systems of differential equations that describe the dynamics of particle systems [3]. Calogero first worked on the problem of describing quantum particles in a one-dimensional system with inverse square pair potentials in 1971 and used his formula to deduce that particles behave elastically in a quantum scattering problem [4]. Later on, Marchioro established a classical variant to Calogero's work for the "three-body problem," and Moser ultimately proved that the clasical generalization was also integrable by connecting the analogue to Hamiltonian systems [5]. The use of Calogero-Moser systems and their generalizations have addressed problems in mathematics including the Hilbert schemes of surfaces in algebraic geometry; symplectic reflection algebras in deformation theory; Koszul algebras in homological algebra; double affine Hecke algebras and Lie groups in representation theory; and Poisson geometry [6]. Calogero-Moser systems have also found their way into the applications of integrable systems to contemporary mathematical physics. A paper by Olalla A. Castro-Alvaredo and Andreas Fring shows that quantum integrable systems can be used to predict the possibility of generating high harmonics from solid state devices and computing expressions of the conductance of a quantum wire [7]. Other applications of integrable systems are fruitful. They are used to model wave phenomena in hydrodynamics, acoustics, nonlinear optics, and plasma physics and have applications in describing the energy transport along proteins, ecological predator-prey equations, charge density waves in organic conductors and electronics network equations [8].

In addition, quasiinvariant polynomials can also lead to a better understanding of the representations of rational Cherednik algebras [4]. This connection has generated a lot of interest from mathematicians in studying quasiinvariants and its corresponding Hilbert series. The spaces of quasiinvariant polynomials of symmetric groups were extensively studied by Feigin and Veselov in 2001 [9], and Felder and Veselov soon computed their Hilbert series in fields of characteristic 0 [2]. Braverman, Etingof, and Finkelberg studied quasiinvariant polynomials twisted by monomial factors and computed the Hilbert series for a wide range of cases in 2016 [10]. Just last year, Ren discovered results relating to quasiinvariants twisted by arbitrary smooth functions and proved that there can only be a finite number of primes p where the Hilbert series in the field of characteristic p is greater than in characteristic 0. Part of our research goal is to extend on the properties of quasiinvariant polynomials in the prime field.

In Section 2, we investigate the space of quasiinvariant polynomials as a module over the ring of symmetric polynomials. We define, characterize, and explain how we use the Hilbert series (an infinite series) to evaluate the size and structure of the space of quasiinvariant polynomials.

In Section 3, we investigate the space of quasiinvariants divisible by a generic homogeneous polynomial. We compute the Hilbert series of the polynomials using two defined operators on Q_m that maintain the invariance of the Hilbert series, generalizing Ren's work when the generic polynomial is x^k [11]. Furthermore, we conjecture a possible method using a geometric interpretation involving n-dimensional spaces.

In Section 4, we prove a theorem regarding sufficient conditions for the Hilbert series in characteristic p to be greater than in characteristic 0. Here, we make use of Felder and Veselov's computation of the Hilbert series in the fields of characteristic 0. In addition, we analyze the Hilbert series when $n \geq 3$, supporting our conjectures with data from our computer programs for the $n = 3$ and $n = 4$ cases.

In Section 5, we give a concluding discussion about our results and give future prospects of our work.

2 Definitions and Background

2.1 Quasiinvariant Polynomials

Quasiinvariant polynomials are generally studied for the action of a finite Coxeter group W in a complex vector space, and the representations of rational Cherednik algebras are usually used to study this space [12]. In our paper, we deal with quasiinvariants when W is the symmetric group S_n.

Definition 2.1. Let k be a field, and let n be a positive integer and m be a non-negative integer. A polynomial $F \in k[x_1, x_2, \ldots, x_n]$ is a *m-quasiinvariant polynomial* if

$$(x_i - x_j)^{2m+1} \mid (1 - s_{i,j})F(x_1, x_2, \ldots, x_n)$$

for all $1 \leq i < j \leq n$. We will denote this polynomial space as Q_m. Morever, we will let $Q_{m,d}$ be the space of *homogeneous* m-quasiinvariant polynomials of degree d.

For example, $Q_0 = k[x_1, x_2, \ldots, x_n]$. An equivalent condition for F to be a m-quasiinvariant polynomial is that

$$\frac{(1 - s_{i,j})F(x_1, x_2, \ldots, x_n)}{(x_i - x_j)^{2m+1}}$$

is a *smooth* function for all $1 \leq i < j \leq n$.

Theorem 2.1. *Q_m is a module over the ring of symmetric polynomials.*

Proof. Let R denote the ring of symmetric polynomials. We claim that $r*q \in Q_m$ for all $r \in R$ and $q \in Q_m$. Here the $*$ operation is polynomial multiplication. Since

$$(1 - s_{i,j})(rq)$$

$$= rq - s_{i,j}(rq)$$

$$= r(1 - s_{i,j})(q)$$

which is divisible by $(x_i - x_j)^{2m+1}$ for all $1 \leq i < j \leq n$, rq is a m-quasiinvariant polynomial. Thus, we can define the operation $* : R \times M \to M$ to be polynomial multiplication. The module conditions are easily verified. $\square$

Theorem 2.2. *Q_m is a finitely generated module.*

Proof. As any field k is a Noetherian ring, the symmetric ring R can be viewed as a polynomial ring over k generated by the elementary symmetric polynomials, i.e.

$$R = k[e_0, e_1, \ldots, e_n],$$

where

$$e_k = \sum_{1 \leq i_1 < i_2 \ldots < i_k \leq n} x_{i_1} \cdots x_{i_k}$$

for $0 \leq k \leq n$. By *Hilbert's Basis Theorem*, R is Noetherian.

Thus, from the *Chevalley-Shepard-Todd* theorem, $k[x_1, x_2, \ldots, x_n]$ is a free module over R finitely generated by homogeneous polynomials $\sigma_1, \sigma_2, \ldots, \sigma_r$ [4-5]. Since Q_m is a submodule of $k[x_1, x_2, \ldots, x_n]$, it is a finitely generated module over R as well. □

2.2 Hilbert Series

R will be used to denote an arbitrary ring in this section.

Definition 2.2. A *graded ring* is a ring R that is a direct sum of abelian groups R_i with the property that

$$R_i R_j \subseteq R_{i+j}$$

for $i, j \in \mathbb{N}_0$

Definition 2.3. A *graded module* is a left module M over a graded ring R such that

$$M = \bigoplus_{i \in \mathbb{N}_0} M_i,$$

and

$$R_i M_j \subseteq M_{i+j}.$$

We can now use a special formal power series to encapsulate the structure and size of M.

Definition 2.4. The *Hilbert series* of a graded module M is

$$HS(t) = \sum_{d \geq 0} \dim_k M_d t^d.$$

Remark. The coefficients are the dimensions of the k-vector spaces M_d.

Since both Q_m and the ring of symmetric polynomials have an increasing filtration by the degree of polynomials, one may define the corresponding Hilbert series for Q_m.

Definition 2.5. The Hilbert series of the space of m-quasiinvariant polynomials is

$$HS_m(t) = \sum_{d \geq 0} \dim_k Q_{m,d} t^d.$$

Remark. Here, $Q_{m,d}$ is viewed as a polynomial vector space.

The remarkable property of Q_m being finitely generated allows the Hilbert series to be written in a condensed form.

Theorem 2.3. *$HS_m(t)$ can be written as*

$$HS_m(t) = \frac{P(t)}{\prod_{i=1}^{n}(1-t^i)}$$

where $P(t)$ is a polynomial with integer coefficients.

Proof. Let set E denote the polynomial basis of Q_m. Let $P(t) = \sum_{k\geq 0} a_k x^k$, where a_k is the number of polynomials in E of degree k. Every degree d polynomial in the basis of $Q_{m,d}$ is formed by the multiplication of a degree l in E and a symmetric polynomial of degree $d-l$. The dimension of the vector space of symmetric polynomials of degree $d-l$ is equal to the number of partitions of $d-l$, since the term $x_1^{i_1} \cdots x_{d-l}^{i_{d-l}}$, where $1 \leq i_1 < i_2 < \cdots < i_{d-l} \leq d-l$ and $\sum_{k=1}^{d-l} i_k = d-l$, is representative of the symmetric polynomial basis element.

It is well know from reflecting Ferrers diagrams that the number of partitions of a number μ equals the number of μ-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_\mu)$ with $\lambda_1, \lambda_2, \ldots, \lambda_\mu \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^{n} i\lambda_i = \mu$. Thus, the coefficient of t^d in

$$P(t)(1+t+t^2+\ldots)(1+t^2+t^4+\ldots)\cdots(1+t^n+t^{2n}+\ldots) = \frac{P(t)}{\prod_{i=1}^{n}(1-t^i)}$$

is the dimension of $Q_{m,d}$.

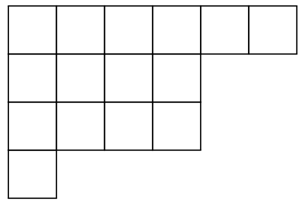
$\square$

Remark. The polynomial $P(t)$ is referred to as the *Hilbert polynomial.* In defining the Hilbert series for graded modules in general, the positive coefficients represent the number of generators of a particular degree. However, if there is some relation amongst the generators, i.e. when the module is not free, then some of the basis elements are extraneous, so this leads to the existence of negative terms in the Hilbert polynomial.

Felder and Veselov computed the formula for the Hilbert series of m-quasiinvariants in the fields of characteristic 0. They used Kirillov's formula in representation theory when the group $G = S_n$ is on the permutation of letters generated by reflections over the hyperplanes $x_i = x_j$ with $i < j$ of $\mathbb{R}^n$ [5]. It is useful to note that their formula depends on the properties of Young diagrams.

Definition 2.6. For a non-negative integer n, let λ denote the partition of integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The *Young diagram* of λ is a collection of empty boxes with k rows that are left-justified so that row i has λ_i boxes.

The following example is a Young diagram the partition (6, 4, 4, 1).



Theorem 2.4. *[5] In a Young diagram with n boxes, let a_k be the number of boxes to the right of the kth box, l_k be the number of boxes below the kth box, and h_k be the hook-length, $a_k + l_k + 1$. Enumerating over all the Young diagrams, the formula for the Hilbert series of Q_m is given by*

$$HS_m(t) = n!t^{m\binom{n}{2}} \sum_{\text{Young diagrams}} \prod_{k=1}^{n} t^{m(l_k-a_k)+l_k} \frac{1-t^k}{h_k(1-t^{h_k})}$$

in fields of characteristic 0. The lowest degree non-trivial polynomials in Q_m have a degree of $mn+1$ by evaluating the Hilbert polynomial. They form an $(n-1)$-dimensional simple S_n-module corresponding to the minimal leg-length, maximal arm-length partition $(n-1, 1, 0, \ldots, 0)$.

Remark. The Hilbert polynomial has the form $1 + (n-1)t^{mn+1} + \ldots$ where the monomial terms are sorted in increasing order of exponents. This implies that every polynomial F in Q_m that has degree $\leq mn$ is symmetric.

Corollary 2.1. *For $n = 2$, the Hilbert series for Q_m in fields of characteristic 0 is*

$$HS_m(t) = \frac{1+t^{2m+1}}{(1-t)(1-t^2)}.$$

Remark. The Hilbert polynomial indicates that the non-trivial generator of Q_m has a degree of $2m+1$. It is also a free module because the coefficients of the Hilbert polynomial are all positive.

3 Quasiinvariant Polynomials Divisible by a Homogeneous Polynomial

Braverman, Etingof, and Finkelberg studied the space quasiinvariant polynomials twisted by monomial factors $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ for $a_1, a_2, \ldots, a_n \in \mathbb{C}$ [11]. Our paper studies a similar analogue to this, with the quasiinvariant space twisted by homogeneous polynomials.

In the field of complex numbers, we will investigate what happens when the space of quasiinvariant polynomials has the imposed condition of being divisible by a generic homogeneous polynomial $g \in \mathbb{C}[x_1, x_2, \ldots, x_n]$. Denote this space by $Q_m(g)$ and the Hilbert series for this space by $HSH_m(g)$. We focus mostly on $n = 2$, letting $x = x_1$, $y = x_2$, and $g = g(x, y)$,

In general, a submodule of a free module is not free, so the Hilbert polynomial in the Hilbert series of this space may have negative terms.

We will start off with a few propositions.

Proposition 3.1. *If $g(x, y) = r(x, y)g_1(x, y)$, where $r(x, y)$ is a symmetric polynomial, then we have*

$$HSH_m(g) = t^{\deg(r)}HSH_m(g_1)$$

Proof. The proof is analogous to that of Theorem 2.2. The symmetric polynomial can be factored out, so every power in the infinite series $HSH_m(g_1)$ is increased by the degree of r to get $HSH_m(g)$ □

Proposition 3.2. *If $g(x, y) = (x - y)^k g_1(x, y)$ for $k \in \mathbb{N}$, where $(x - y) \nmid g_1(x, y)$, then*

$$HSH_m(g) = t^k HSH_{m-\frac{k}{2}}(g_1)$$

if $k \leq 2m$ is even, and

$$HSH_m(g) = t^k HSH_{m-\frac{k+1}{2}}(g_1)$$

if $k \leq 2m - 1$ is odd.

Proof. When k is even, $(x - y)^k$ is invariant under the $s_{i,j}$ operator, implying that it is a symmetric polynomial. The result follows from Proposition 3.1

When $k = 2j + 1$ is odd, we just need $g_1(x, y)f(x, y) + g_1(y, x)f(y, x)$ to be divisible by $(x - y)^{2(m-j)}$ for $f(x, y) \in \mathbb{C}[x_1, x_2]$ such that $f(x, y)g(x, y) \in Q_m(g)$. From this, by setting $x = y$, we get $g_1(x, x)f(x, x) = 0 \implies f(x, y) = (x - y)f_1(x, y)$. The relation reduces to $f_1(x, y) - f_1(y, x)$ being divisible by $(x - y)^{2(m-j-1)+1}$, so we can conclude that

$$HSH_m(g) = t^k HSH_{m-\frac{k+1}{2}}(g_1)$$

$\square$

Proposition 3.3. *If $g \in Q_{m_1}$, where $m_1 \geq m$, then*

$$HSH_m(t) = t^{\deg(g)} HS_{\frac{2m-k}{2}}(t).$$

Proof. Note that $g(x,y)f(x,y) - g(y,x)f(y,x) = (g(x,y) - g(y,x))f(x,y) + g(y,x)(f(x,y) - f(y,x))$. Letting k be the highest power of $x - y$ dividing g, we just want $f(x,y) - f(y,x)$ to be divisible by $(x-y)^{2m+1-k}$. Using the fact that the highest power of $x - y$ dividing $p(x,y) - p(y,x)$ is always odd for any p, we can see that

$$HSH_m(t) = t^{deg(g)} HS_{\frac{2m-k}{2}}(t).$$

$\square$

Next, we will generalize a specific result of [4] to compute the Hilbert series.

Theorem 3.1. *For $k \in \mathbb{N}$ and $\alpha_1, \alpha_2 \in \mathbb{C}$, if $g(x,y) = \alpha_1 x^k + \alpha_2 y^k$ and $\alpha_1^2 \neq \alpha_2^2$, then*

$$HSH_m(g) = t^k \left(\frac{t^{2m} + t^{2m+1} + \sum_{i=1}^{m} t^{2(m-i)+min(i,k)} - \sum_{i=1}^{m} t^{2(m+1-i)+min(i,k)}}{(1-t)(1-t^2)} \right)$$

.

Proof. Let $M : \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ be the operator defined as $M\left[f(x,y)\right] = \alpha_1 f(x,y) - \alpha_2 f(y,x)$. We claim that for polynomials $f(x,y) \in \dfrac{1}{g(x,y)} Q_m(g)$, $M\left[f(x,y)\right] \in \dfrac{1}{x^k} Q_m(x^k)$. Since

$$\begin{aligned}
&\frac{(1-s_{1,2})g(x,y)f(x,y)}{(x-y)^{2m+1}} \\
&= \frac{(\alpha_1 x^k + \alpha_2 y^k)f(x,y) - (\alpha_1 y^k + \alpha_2 x^k)f(y,x)}{(x-y)^{2m+1}} \\
&= \frac{x^k(\alpha_1 f(x,y) - \alpha_2 f(y,x)) - y^k(\alpha_1 f(y,x) - \alpha_2 f(x,y))}{(x-y)^{2m+1}}. \\
&= \frac{(1-s_{1,2})x^k(\alpha_1 f(x,y) - \alpha_2 f(y,x))}{(x-y)^{2m+1}}
\end{aligned}$$

is smooth, $M[f] \in \dfrac{1}{x^k} Q_m(x^k)$.

Moreover, we will prove that M is bijective by explicitly constructing its inverse operator. We can independently work with the homogeneous components. For a non-negative integer

d, let $f(x,y) = \sum_{i=0}^{d} a_i x^i y^{d-i}$ and $h(x,y) = \sum_{i=0}^{d} b_i x^i y^{d-i}$ with $a_i, b_i \in \mathbb{C}$ for $0 \le i \le d$. Setting $\alpha_1 f(x,y) - \alpha_2 f(y,x)$ equal to $h(x,y)$, we get

$$\alpha_1 f(x,y) - \alpha_2 f(y,x)$$

$$= \sum_{i=0}^{d} (\alpha_1 a_i - \alpha_2 a_{d-i}) x^i y^{d-i} = \sum_{i=0}^{d} b_i x^i y^{d-i}.$$

The coefficients in f are given by $a_i = \dfrac{(\alpha_1 b_i + \alpha_2 b_{d-i})}{(\alpha_1^2 - \alpha_2^2)}$ for $0 \le i \le d$, and this mapping is valid. Hence, M is bijective, so there is a bijection between $f \in \dfrac{1}{g(x,y)} Q_m(g)$ and $h \in \dfrac{1}{x^k} Q_m(x^k)$. From [4],

$$HSH_m(g) = HSH_m(x^k)$$
$$= t^k \left(\frac{t^{2m} + t^{2m+1} + \sum_{i=1}^{m} t^{2(m-i)+min(i,k)} - \sum_{i=1}^{m} t^{2(m+1-i)+min(i,k)}}{(1-t)(1-t^2)} \right)$$

. ☐

Remark. Let $k = 1$ be the degree of g. Then, with casework on the values of m, the Hilbert series of $Q_m(\alpha_1 x + \alpha_2 y)$ is

$$t \left(\frac{t^{2m} + t^{2m+1} + \sum_{i=1}^{m} t^{2(m-i)+min(i,1)} - \sum_{i=1}^{m} t^{2(m+1-i)+min(i,1)}}{(1-t)(1-t^2)} \right)$$

$$= t \left(\frac{t^{min(m,1)}(1 + t^{max(1,2(m-1)+1)})}{(1-t)(1-t^2)} \right).$$

Definition 3.1. For $\mathrm{d} \in \mathbb{C} \setminus \{-1, 1\}$ and $f(x,y) \in \mathbb{C}[x,y]$, let $\mathrm{L^d}$ be the operator $\mathrm{L^d}$: $\mathbb{C}[x,y] \to \mathbb{C}[x,y]$ defined by

$$\mathrm{L^d}[f(x,y)] = f(x + \mathrm{d}y, y + \mathrm{d}x).$$

Remark. $\mathrm{L^d}$ is bijective and the inverse operator is given by

$$(\mathrm{L^d})^{-1}[f(x,y)] = f\left(\frac{x - \mathrm{d}y}{1 - \mathrm{d}^2}, \frac{y - \mathrm{d}x}{1 - \mathrm{d}^2} \right)$$

Lemma 3.1. *For* $\mathrm{d} \in \mathbb{C} \setminus \{-1, 1\}$ *and a generic homogeneous polynomial* $g(x,y) \in \mathbb{C}[x_1, x_2]$,

$$HSH_m(g) = HSH_m(\mathrm{L^d}[g]).$$

Proof. We claim that for every polynomial $f_1 \in Q_m(g)$, $\mathrm{L}^{\mathrm{d}}[f_1] \in Q_m(\mathrm{L}^{\mathrm{d}}[g])$.

$$\mathrm{L}^{\mathrm{d}}\left[\frac{f_1(x,y)-f_1(y,x)}{(x-y)^{2m+1}}\right]$$
$$=\frac{f_1(x+\mathrm{d}y,y+\mathrm{d}x)-f_1(y+\mathrm{d}x,x+\mathrm{d}y)}{(1-\mathrm{d})(x-y)}$$
$$=\frac{1}{1-\mathrm{d}}\frac{f_2(x,y)-f_2(y,x)}{(x-y)^{2m+1}}$$

is smooth, where $f_2(x,y)=\mathrm{L}^{\mathrm{d}}[f_1(x,y)]$. The quasiinvariant condition is unaffected by the constant factor, so $f_2 \in Q_m(\mathrm{L}^{\mathrm{d}}[g])$. Thus, the bijection between $Q_m(g)$ and $Q_m(\mathrm{L}^{\mathrm{d}}[g])$ implies that

$$HSH_m(g)=HSH_m(\mathrm{L}^{\mathrm{d}}[g]).$$

□

Now, we are ready to prove a main result of this section.

Theorem 3.2. *Let $\alpha_1,\alpha_2,\alpha_3 \in \mathbb{C}$. For a degree 2 generic polynomial $g=\alpha_1x^2+\alpha_2xy+\alpha_3y^2$, then*

$$HSH_m(g)=\begin{cases}\dfrac{t^2+t^{2m+3}}{(1-t)(1-t^2)} & \alpha_1=\alpha_3\\ t^2\left(\dfrac{t^{min(m-1,1)}(1+t^{max(1,2(m-2)+1)})}{(1-t)(1-t^2)}\right) & \alpha_1+\alpha_2+\alpha_3=0,\alpha_1\neq\alpha_3\\ t^2\left(\dfrac{t^{min(m,1)}(1+t^{max(1,2(m-1)+1)})}{(1-t)(1-t^2)}\right) & \alpha_1-\alpha_2+\alpha_3=0,\alpha_1\neq\alpha_3\\ t^2\left(\dfrac{t^{2m}+t^{2m+1}+\sum_{i=1}^{m}t^{2(m-i)+min(i,2)}-\sum_{i=1}^{m}t^{2(m+1-i)+min(i,2)}}{(1-t)(1-t^2)}\right) & otherwise\end{cases}$$

Proof. By using Lemma 3.1, we will choose a suitable d such that the coefficient of xy in $\mathrm{L}^{\mathrm{d}}[g]$ is 0. The motivation for this is to transform g into the form in Theorem 3.1. Write $g(x,y)$ as $C_1(x-\beta_1y)(x-\beta_2y)$ for some C_1,β_1,β_2 in $\mathbb{C}$. Applying the operator, we get

$$\mathrm{L}^{\mathrm{d}}[C_1(x-\beta_1y)(x-\beta_2y)]$$
$$=C_2\left(x-\frac{\beta_1-\mathrm{d}}{1-\beta_1\mathrm{d}y}\right)\left(x-\frac{\beta_2-\mathrm{d}}{1-\beta_2\mathrm{d}}y\right)$$

for some $C_2 \in \mathbb{C}$. However, note that when $\beta_1\beta_2 = 1$, $g(x, y)$ is a symmetric polynomial, so $\alpha_1 = \alpha_3$. By Proposition 3.1, $HSH_m(g) = \dfrac{t^2 + t^{2m+3}}{(1-t)(1-t^2)}$.

In addition, if $\beta_1 = 1$, then β_2 must be -1 to make the coefficient of xy to be 0, but this is not the form in Theorem 3.1. Thus, when $\beta_1 = 1$ and g is not a symmetric polynomial, using Proposition 3.2 and Theorem 3.1 results in

$$HSH_m(g) = tHSH_{m-1}(x) = t^2 \left(\frac{t^{min(m-1,1)}(1 + t^{max(1,2(m-2)+1)})}{(1-t)(1-t^2)} \right)$$

.

Similarly, when $\beta_1 = -1$, β_2 is 1 to make the coefficient of xy equal 0, but this does not result in the form in Theorem 3.1. So when $\beta_1 = -1$ and g is not a symmetric polynomial, using Proposition 3.1 and Theorem 3.1 yields

$$HSH_m(g) = tHSH_m(g) = t^2 \left(\frac{t^{min(m,1)}(1 + t^{max(1,2(m-1)+1)})}{(1-t)(1-t^2)} \right)$$

.

In all other cases, the suitable d transforms g into the form in Theorem 3.1. $\square$

If we treat the coefficients $\alpha_1, \alpha_2, \alpha_3$ as coordinates of $\mathbb{R}^3$, then we can view R^3 as being partitioned into several multi-dimensional spaces, where each space is associated with a Hilbert series. In particular, Theorem 3.1 shows $\mathbb{R}^3$ being partitioned into $1, 2,$ and 3-dimensional spaces. For $n \geq 3$, we hope to investigate more on the geometric nature of this observation.

For a generic homogeneous $g(x, y) \in \mathbb{C}[x, y]$, computer programs show that the Hilbert series takes a form depending on the quasiinvariance of g, having certain divisibility conditions.

Conjecture 3.1. *Suppose $g(x, y)$ is not divisible by a symmetric polynomial and $(x - y)$. Let m_1 be the maximum integer such that*

$$(x-y)^{2m_1+1} | g(x, y) - g(y, x)$$

Then,

$$HSH_m(g) = \begin{cases} t^{\deg(g)} \left(\dfrac{1+t^{2m+1}}{(1-t)(1-t^2)} \right) & m \leq m_1 \\ t^{\deg(g)} \left(\dfrac{t^{2m}+t^{2m+1}+\sum\limits_{i=1}^{m} t^{2(m-i)+min(i,\deg(g))} - \sum\limits_{i=1}^{m} t^{2(m+1-i)+min(i,\deg(g))}}{(1-t)(1-t^2)} \right) & m > m_1 \end{cases}$$

Remark. This is supported by computer programs calculations when computing the Hilbert series for hundreds of different $g(x,y)$. For example, when $g(x,y) = x^4 - 2x^3y$, m_1 is 1.

$$HSH_m(g) = t^4 \left(\frac{1+t^{2m+1}}{(1-t)(1-t^2)} \right)$$

when $m \leq m_1$ and

$$HSH_m(g) = t^4 \left(\frac{t^{2m}+t^{2m+1}+\sum_{i=1}^{m} t^{2(m-i)+min(i,4)} - \sum_{i=1}^{m} t^{2(m+1-i)+min(i,4)}}{(1-t)(1-t^2)} \right)$$

when $m > m_1$. We suspect that we can find an isomorphism between the space divisible by $g(x,y)$ and $x^{\deg(\mathrm{g})}$ for $m > m_1$, a combination of the methods used in Theorem 3.1 and Lemma 3.1.

4 m-Quasiinvariant Polynomials in $\mathbb{C}$ and $\mathbb{F}_p$

Ren previously studied quasiinvariant polynomials and its Hilbert series in the fields of characteristic p [11]. Two of his results are listed below.

Theorem 4.1. *[11] For each m, there are only finitely many primes p for which the Hilbert series of Q_m is greater in $\mathbb{F}_p$ than in $\mathbb{C}$.*

Theorem 4.2. *[11] When $n = 2$, the Hilbert series for Q_m over characteristic p coincides with that of characteristic 0. It is $\frac{1+t^{2m+1}}{(1-t)(1-t^2)}$ in all fields.*

Theorem 4.2 solves the problem in the case when $n = 2$. In our major result, we prove sufficient conditions for the Hilbert series to be greater in the prime field than in the field of complex numbers when $n \geq 3$.

Theorem 4.3. *Let p be a prime number. Let $a \geq 0$, $n \geq 3$, $m \geq 0$, and $k \geq 0$ be integers. Moreover, let $b \geq 0$ be a half-integer if $p = 2$ and an integer if p is odd. If*

$$p^a(nk+1) + 2b\binom{n}{2} \leq mn$$

and

$$p^a(2k+1) + 2b \geq 2m+1,$$

then the Hilbert series of Q_m in n variables is greater in $\mathbb{F}_p$ than in $\mathbb{C}$

Proof. We wish to show that the minimal degree of a non-symmetric generator of Q_m in $\mathbb{F}_p$ is greater than the one in $\mathbb{C}$. This will imply that the Hilbert series is greater in $\mathbb{F}_p$ by comparing the coefficients in increasing degree.

From Felder and Veselov's formula, we can see that the Hilbert polynomial is of the form $1+(n-1)t^{mn+1}+...$, where the monomial terms are sorted in increasing order of exponents. Thus, the non-zero coefficient in of t^{mn+1} implies the existence of a non-symmetric generator P_m of degree $mn+1$. Consider the polynomial

$$G(x_1, x_2, ...x_n) = P_k^{p^a} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2b}.$$

We will show that $(x_i - x_j)^{2m+1}$ divides $(1 - s_{i,j})G$ and $\deg(G) \leq mn$ in $\mathbb{F}_p$ for all $1 \leq i < j \leq n$, meaning there is a positive coefficient in front of $t^{\deg G}$ which will prove the theorem.

Note that

$$(1 - s_{i,j})G = (1 - s_{i,j})(P_k^{p^a}) \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2b},$$

where we used the fact that $\prod_{1\le i<j\le n}(x_i-x_j)^{2b}$ is a symmetric polynomial.

From the *Frobenius endomorphism*, $(1-s_{i,j})(P_k^{p^a}) = ((1-s_{i,j})P_k)^{p^a})$. This implies that the power of $(x_i - x_j)$ dividing $(1-s_{i,j})G$ is at least $(2k+1)p^a + 2b$.

Furthermore, $\deg(G) = (nk+1) + 2b\binom{n}{2} \le mn$, so the existence of the minimal non-symmetric polynomial G is confirmed. □

For the cases $n = 3$ and $n = 4$, computer calculations show that the conditions in Theorem 4 seem to be necessary as well. See Figure 1.

Conjecture 4.1. *If the Hilbert series of Q_m in n variables is greater in $\mathbb{F}_p$ than in $\mathbb{C}$, then there exist variables a, b, k defined in Theorem 4 such that*

$$p^a(nk+1) + 2b\tbinom{n}{2} \le mn$$
$$p^a(2k+1) + 2b \ge 2m+1,$$

Furthermore, the minimal non-symmetric polynomial that generates Q_m is of the form

$$G = P_k^{p^a}\prod_{1\le i<j\le n}(x_i-x_j)^{2b}.$$

Remark. When $p > mn$, this conjecture implies that the Hilbert series are the same. From Theorem 4.1, we suspect that all the primes where the Hilbert series is greater is bounded by mn.

In the two tables below, the squares are colored if the Hilbert series is greater in $\mathbb{C}$ than in $\mathbb{F}_p$ as a function of m and p. As every entry in the following tables match the conjecture, we highly suspect that the claim is true. For the $n = 3$ table in Figure 1, we computed up to $p \le 37$ and $m \le 11$ and for the $n = 4$ table in Figure 2, we computed $p \le 37$ and $m \le 9$. Additionally, the data highly supports Conjecture 4.1 that the minimal non-symmetric polynomial generator is of that form. The different colors are indicative of the different values of a and k that make the condition hold true. $a = 1, k = 0$ is represented by the blue color, $a = 1, k = 1$ is represented by red, $a = 1, k = 2$ is represented by yellow, $a = 2, k = 0$ is represented by green, $a = 3, k = 0$ is represented by magenta, $a = 4, k = 0$ is represented by black, and $a = 5, k = 0$ is represented by cyan.

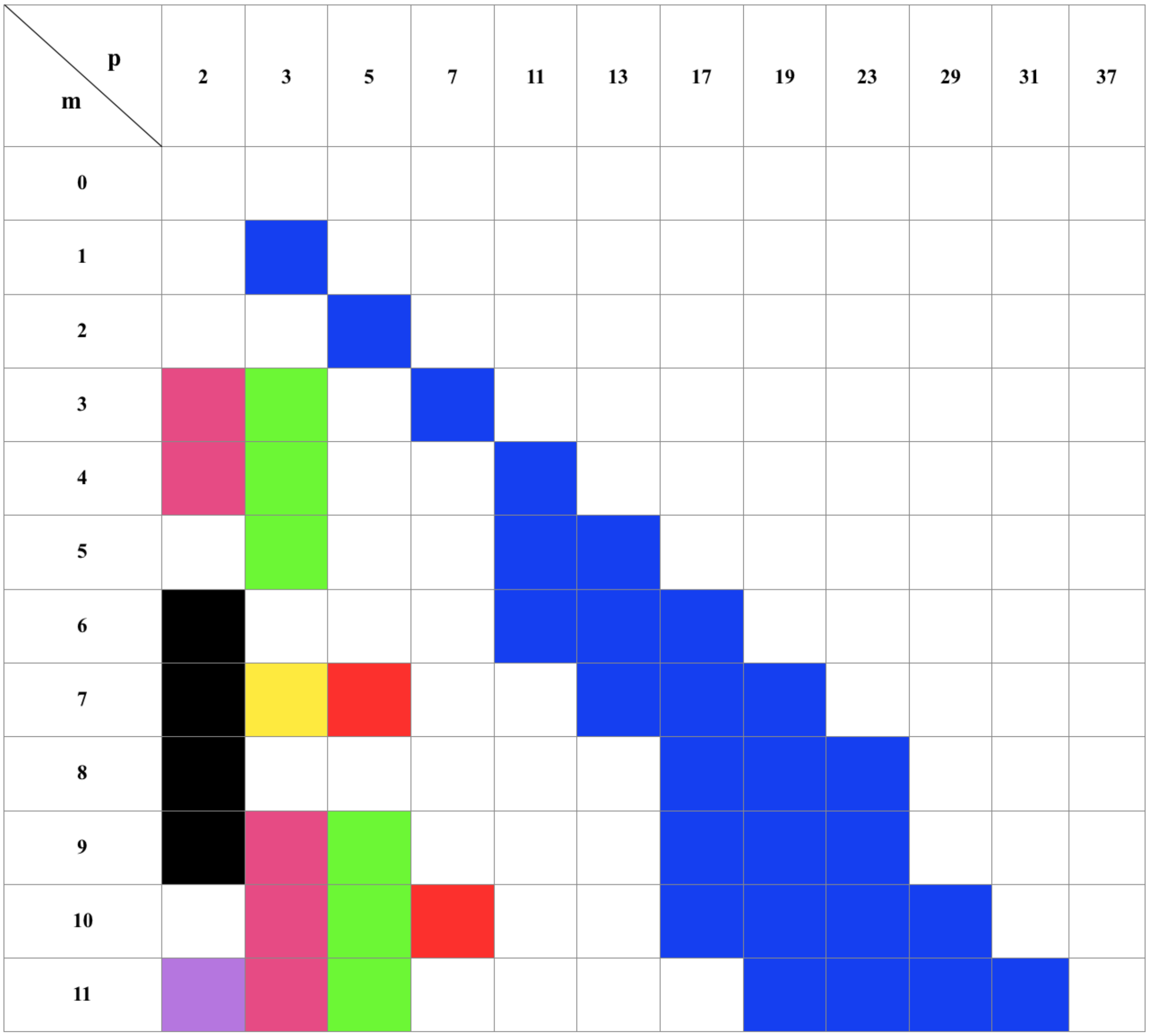


Figure 1: $n = 3$

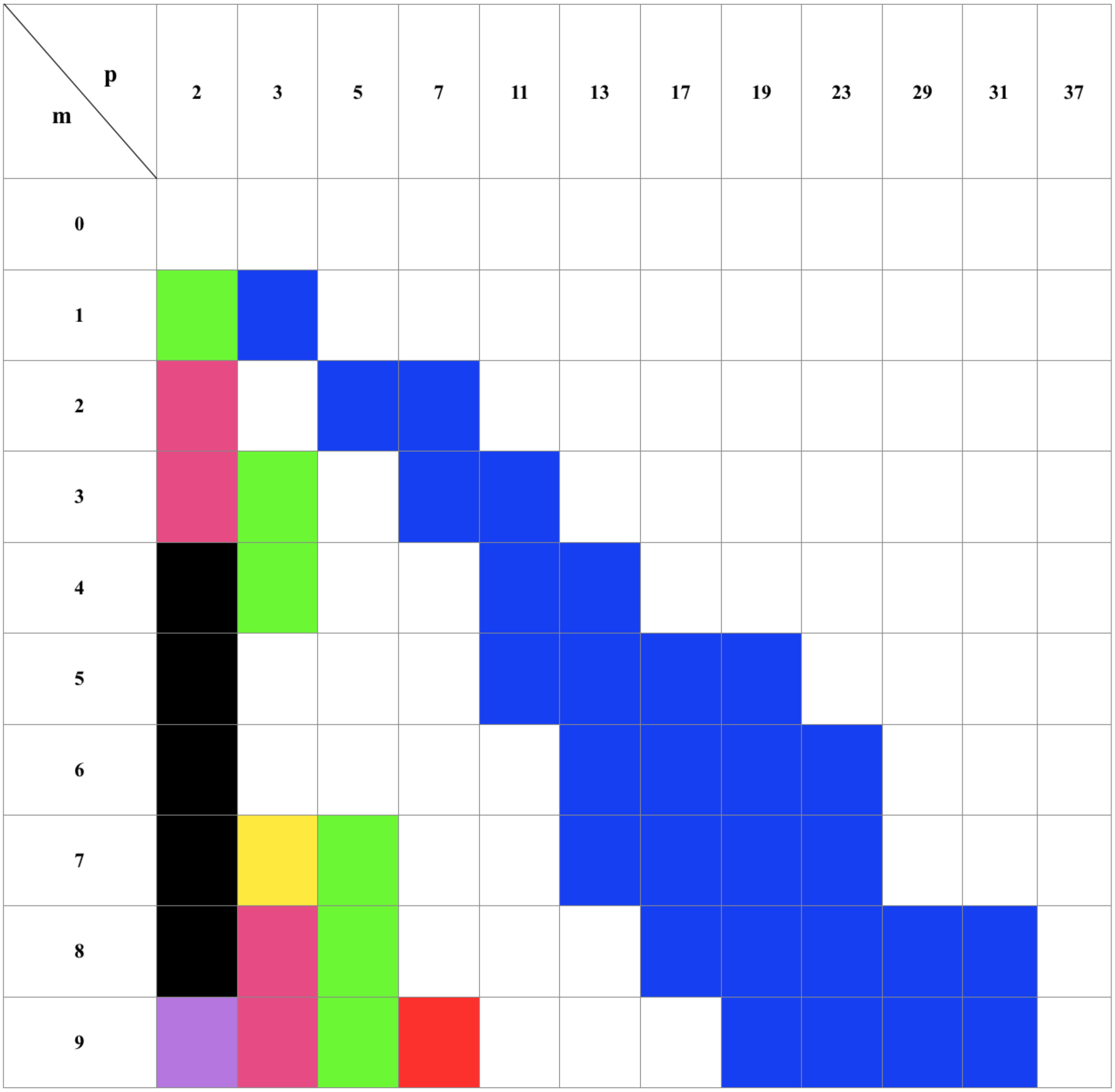


Figure 2: $n = 4$

5 Conclusion

In this paper, we have computed the Hilbert series for m-quasiinvariant spaces in the $n = 2$ case. We also found the sufficient conditions for the Hilbert series of the m-quasiinvariants to be greater in the field of characteristic p than in characteristic 0. We believe the proof for $n \geq 3$ cases requires showing the existence of a quasiinvariant polynomial in $\mathbb{C}$ that reduces to a quasiinvariant polynomial in $\mathbb{F}_p$. For a better understanding and to prove generalizations, modular representation theory of S_n may be used to find out how the Cherednik algebra acts on the space of quasiinvariants.

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