ARITHMETIC PROPERTIES OF WEIGHTED CATALAN NUMBERS

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Definition

The Catalan numbers are the sequence of integers

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

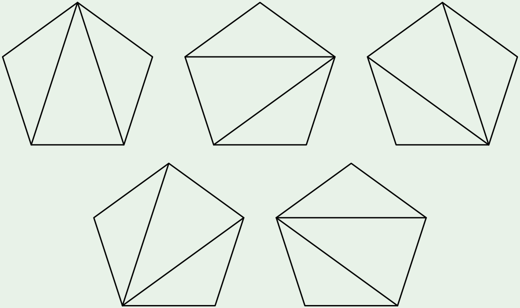
The first few values are:

$$\begin{aligned} C_0 &= 1 & C_3 &= 5 & C_6 &= 132 \\ C_1 &= 1 & C_4 &= 14 & C_7 &= 429 \\ C_2 &= 2 & C_5 &= 42 & C_8 &= 1430. \end{aligned}$$

C_n is the number of full triangulations of an $(n+2)$-gon.

Example

$$C_3 = 5$$



C_n is the number of ways to pair n sets of brackets.

Example

$$C_3 = 5$$

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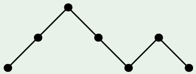
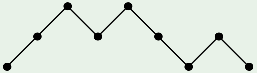
Definition

A Dyck path of length $2n$ is a continuous broken line lying in the first quadrant of the plane, starting at the origin $(0, 0)$ and consisting of n “up-steps” in the direction $\langle 1, 1\rangle$ and n “down-steps” in the direction $\langle 1, -1\rangle$.

Example

length 6: 

length 8: 

C_n is the number of Dyck paths of length $2n$.

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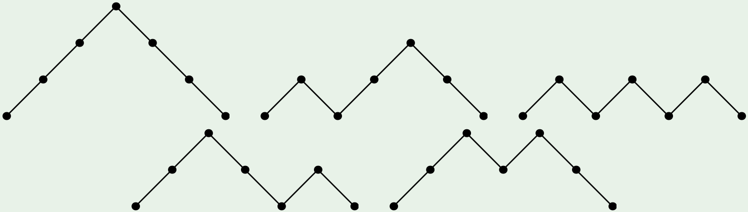

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From the recurrence, we get the equation

$$\mathcal{C}(x) = x \cdot \mathcal{C}(x)^2 + 1,$$

where $\mathcal{C}(x)$ is the generating function of the Catalan numbers:

$$C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots .$$

Solving $\mathcal{C}(x) = x \cdot \mathcal{C}(x)^2 + 1$ gives

$$\mathcal{C}(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

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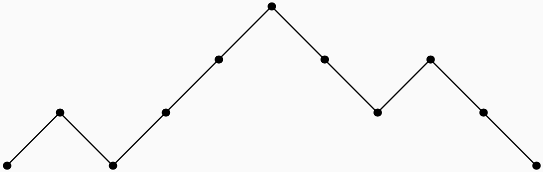
Another expression for $\mathcal{C}(x)$ is the continued fraction:

$$\mathcal{C}(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \ddots}}}}.$$

Suppose we have a sequence of integers $\mathbf{b} = (b_0, b_1, b_2, b_3, \dots)$ and a Dyck path P, as shown



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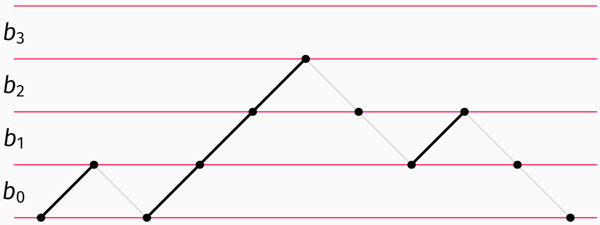

$$\mathrm{wt}(P) = b_0^2 b_1^2 b_2$$

Definition

The *nth weighted Catalan number* $C_n^{\mathbf{b}}$ is the sum of weights over all Dyck paths P of length $2n$.

$$C_n^{\mathbf{b}} = \sum_{\text{paths } P} \text{wt}(P).$$

The first few values in terms of b_i are:

$$C_0^{\mathbf{b}} = 1$$

$$C_1^{\mathbf{b}} = b_0$$

$$C_2^{\mathbf{b}} = b_0^2 + b_0 b_1$$

$$C_3^{\mathbf{b}} = b_0^3 + 2b_0^2 b_1 + b_0 b_1^2 + b_0 b_1 b_2$$

$$C_4^{\mathbf{b}} = b_0^4 + 3b_0^3 b_1 + 3b_0^2 b_1^2 + b_0 b_1^3 + 2b_0^2 b_1 b_2 + 2b_0 b_1^2 b_2 + b_0 b_1 b_2^2 + b_0 b_1 b_2 b_3$$

$$C_5^{\mathbf{b}} = b_0^5 + 4b_0^4 b_1 + 6b_0^3 b_1^2 + 3b_0^3 b_1 b_2 + 4b_0^2 b_1^3 + 2b_0^2 b_1 b_2^2 + b_0 b_1 b_2^3 + 3b_0 b_1^2 b_2^2 + b_0 b_1 b_2 b_3^2 + 3b_0 b_1^3 b_2 + 2b_0 b_1 b_2^2 b_3 + 2b_0 b_1^2 b_2 b_3 + b_0 b_1 b_2 b_3 b_4$$

$$\vdots$$

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The sequence $\mathbf{b} = (1, 2, 3, 4, \ldots)$ gives $C_n^{\mathbf{b}} = (2n-1)!!$.
Similarly, $\mathbf{b} = (1, 1, 2, 2, 3, 3, 4, \ldots)$ gives $C_n^{\mathbf{b}} = n!$.

Example

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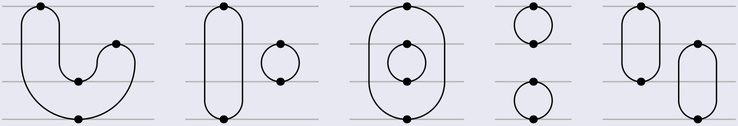
- Macdonald polynomials
- Geometry of Hilbert schemes
- Harmonic analysis
- Representation theory
- Mathematical physics
- Algebraic combinatorics

Theorem (A. Postnikov, 2000)

The number of plane Morse links of order n are the weighted Catalan numbers $C_n^{\mathbf{b}}$ for the sequence $\mathbf{b} = (1^2, 3^2, 5^2, 7^2, \ldots)$. This sequence is commonly denoted L_n.



Theorem (Kummer)

If $\nu_2(n)$ denotes the largest power of 2 dividing n, and $s_2(n)$ denotes the sum of the binary digits of n, then

$$\nu_2(C_n) = s_2(n+1) - 1.$$

Definition

Let Δ be the difference operator, acting on functions $f \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ by $(\Delta f)(x) = f(x+1) - f(x)$.

Theorem (A. Postnikov, 2006)

If the sequence $\mathbf{b}$ *satisfies:*

- $b(0)$ *is odd, and*
- $2^{n+1} \mid (\Delta^n \mathbf{b})(x)$ *for all* $n \geq 1$ *and* $x \in \mathbb{Z}_{\geq 0}$*,*

then

$$\nu_2(C_n^{\mathbf{b}}) = \nu_2(C_n) = s_2(n+1) - 1.$$

Let $\mathcal{S}$ be the shift operator:

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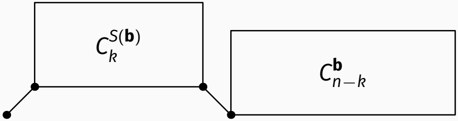
Then we have the recurrence:

$$C^{\mathbf{b}}_{n+1} = b_0 \sum_{k=0}^{n} C_k^{\mathcal{S}(\mathbf{b})} C^{\mathbf{b}}_{n-k}.$$



Recall the generating function for C_n

$$\mathcal{C}(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \ddots}}}.$$

For $C_n^{\mathbf{b}}$,

$$\mathcal{C}^{\mathbf{b}}(x) = \cfrac{1}{1 - \cfrac{b_0 x}{1 - \cfrac{b_1 x}{1 - \cfrac{b_2 x}{1 - \ddots}}}}.$$

We want to study $C_n^{\mathbf{b}}$ modulo primes p, or more generally, p^n.

Since $C_n^{\mathbf{b}}$ is a polynomial in b_i, it suffices to consider the residues of b_i modulo p.

Theorem

Let $\mathbf{b} = (b_0, b_1, b_2, \dots)$ and p be prime. Then $C_n^{\mathbf{b}}$ is eventually periodic modulo p iff any of b_i are congruent to 0 mod p.

This is because if any of the b_i are 0, then $\mathcal{C}^{\mathbf{b}}(x)$ is rational, i.e.

$$\mathcal{C}^{\mathbf{b}}(x) = \frac{p(x)}{q(x)}$$

for some polynomials $p(x)$, $q(x)$.

If $p = 2$, then we can describe the period:

Theorem

Consider a sequence $\mathbf{b} \in \mathbb{F}_2^{\mathbb{N}}$ *such that* $b_k = 0$*, and* $b_i = 1$ *for* $i < k$*. Then*

$$\mathcal{C}^{\mathbf{b}}(x) = \frac{p_{k+1}(x)}{p_{k+2}(x)},$$

where

$$p_{k+2}(x) = p_{k+1}(x) + x \cdot p_k(x),$$

with $p_0(x) = 0$ *and* $p_1(x) = 1$.

The period of the sequence $\mathcal{C}_n^{\mathbf{b}}$ *is equal to the minimal number* m *such that* $p_{k+2}(x)$ *divides* $x^m - 1$.

It turns out that $p_{2^k}(x) = 1$ for all k. In particular, we get:

Corollary

Let $k \geq 1$ be a natural number. Consider a sequence $\mathbf{b} \in \mathbb{F}_2^{\mathbb{N}}$ such that $b_{2^k-2} = 0$, and $b_i = 1$ for $i < 2^k - 2$. Then

$$\mathcal{C}^{\mathbf{b}}(x) = p_{2^k-1}(x).$$

In particular, it is a polynomial of degree $2^{k-1} - 1$ and so the sequence $\mathcal{C}_n^{\mathbf{b}}$ is identically 0 for $n \geq 2^{k-1}$.

- What is the period of L_n modulo p^n?

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The period of L_n modulo 3^{k+3} is exactly $2 \cdot 3^k$.

- What is the period of $C_n^{\mathbf{b}}$ modulo p^n?
- Can we classify sequences for primes greater than 2?

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Questions?