Low Dimensional d-Algebras

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PRIMES Conference

What is an Algebra?

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If F is the field of scalars for an algebra A, then we say that A is an algebra over F.

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- $a \cdot (b + c) = a \cdot b + a \cdot c$
- $k(a \cdot b) = (ka) \cdot b = a \cdot (kb)$

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$$\mathbb{Q}[x]=\{a_0+a_1x+a_2x^2+\cdots+a_nx^n \mid a_0,\ldots,a_n\in\mathbb{Q}\}$$

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Commutative Algebras

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- All three algebras from the previous slide had the property that multiplication was **commutative**. This means that the equation:

  $$a \cdot b = b \cdot a \text{ for all } a, b \in A$$

  is true when $A = \mathbb{C}$ or $A = \mathbb{Q}[x]$ (or any other polynomial algebra).

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$$A \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ but } B \cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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Suppose $a_1, \ldots, a_n$ are all elements of A. A **linear combination** of $a_1, \ldots, a_n$ is a finite sum of the form:

$$k_1 a_1 + k_2 a_2 + \cdots + k_n a_n$$

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For example, set $A = M_{2,2}(\mathbb{R})$ and $F = \mathbb{R}$. Then the matrix $\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$ is a linear combination of the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ because:

$$\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} = 5 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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- As a matter of fact, every matrix in $M_{2,2}(\mathbb{R})$ is a linear combination of the matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

because:

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because:

$$\begin{pmatrix}a & b\\c & d\end{pmatrix} = a\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + d\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

- Likewise, every element of the algebra $\mathbb{C}$ is a linear combination of the numbers 1 and i, because:

$$a + bi = a(1) + b(i)$$

Suppose A is an algebra over F.

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A set of elements $a_1, \ldots, a_n \in A$ is said to **span** A if every element of A can be written as a linear combination of $a_1, \ldots, a_n$.

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The **dimension** of A is the minimum number of elements in A needed to span A. The dimension of A is denoted $\dim(A)$.

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- $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ span $M_{2,2}(\mathbb{R})$

Definition

The **dimension** of A is the minimum number of elements in A needed to span A. The dimension of A is denoted $\dim(A)$.

For example:

- $\dim(\mathbb{C}) = 2$

Dimension

Suppose A is an algebra over F.

Definition

A set of elements $a_1, \ldots, a_n \in A$ is said to **span** A if every element of A can be written as a linear combination of $a_1, \ldots, a_n$.

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- $\dim(M_{2,2}(\mathbb{R})) = 4$

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4. d-commutativity:
$$a \cdot b = b \cdot a + d(b) \cdot d(a)$$

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Quick sidenote: we also must assume that F is algebraically closed and $\mathrm{char}(F) = 2$.

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- Our algebra A has basis (over F) $\{1, v_1, v_2, v_3, w_1, w_2, w_3\}$, and we define:

$$d(1) = d(v_1) = d(v_2) = d(v_3) = 0$$

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- It has multiplication table:

$\cdot$	v_1	v_2	v_3	w_1	w_2	w_3
v_1	0	v_3	0	0	w_3	0
v_2	v_3	0	0	w_3	0	0
v_3	0	0	0	0	0	0
w_1	0	w_3	0	0	v_3	0
w_2	w_3	0	0	0	0	0
w_3	0	0	0	0	0	0

- A is noncommutative because $w_1 \cdot w_2 = v_3$, but $w_2 \cdot w_1 = 0$.

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- Main Result:

Theorem

Up to isomorphism, there exists only one noncommutative d-algebra of dimension 7.

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