Continuum Modelling of Traffic Systems with Autonomous Vehicles

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Abstract

Describing the behavior of automobile traffic via mathematical modeling and computer simulation has been a field of study conducted by mathematicians throughout the last century. One of the oldest models in traffic flow theory casts the problem in terms of densities and fluxes in partial differential conservation laws. In the past few years, the rise of autonomous vehicles (driven by software without human intervention) presents a new problem for classical traffic modeling. Autonomous vehicles react very differently from the traditional human-driven vehicles, resulting in modifications to the underlying partial differential equation constitutive laws. In this paper, we aim to provide insight into some new proposed constitutive laws by using continuum modelling to study traffic flows with a mix of human and autonomous vehicles. We also introduce various existing traffic flow models and present a new model for traffic flow that is based on an interaction between human drivers and autonomous vehicles where each vehicle can only measure the total density of surrounding cars, regardless of human or autonomous status. By implementing the Lax-Friedrichs scheme in Octave, we test how these different constitutive laws perform in our model and analyze the density curves that form over time steps. We also analytically derive and implement a Roe solver for a class of coupled conservation equations in which the velocities of cars are polynomial functions of the total density of surrounding cars regardless of type. We hope that our results could help civil engineers bring forth real progress in implementing efficient road systems that integrates both human-operated and unmanned vehicles.

1 Introduction and Motivation

Traffic flow is the study of how vehicles interact with each other on the road. There have been numerous methods proposed for analyzing traffic behavior. The models can generally be categorized into either microscopic or macroscopic views, each one having its own advantages and disadvantages. Mathematicians such as Greenshields used a velocity function linearly related to density, and mathematicians have subsequently refined his model decades later [1]. With the development of technology came increased computing power, allowing mathematicians to statistically verify their predictions with computer simulations. This project draws upon the ideas of various existing models to form a new model that accounts for both autonomous and human drivers as well. The primary mathematical tool used for modelling traffic microscopically is partial differential equations, PDEs for short, which are one of the strongest tools for mathematically modelling the real world. Our objective for creating a new model is challenging because while there are already many partial differential models of traffic flow, there are not many taking autonomous vehicles into account. Thus, we use a computer Octave simulation and analyze its results.

Aiming for smoother and more accurate density curves in our simulations, we tried to adopt a Roe solver, an approximation Riemann solver derived by mathematician

Phil Roe [2]. Coding a working Roe solver, we drew remarkable conclusions from our simulations. We found that the Roe solver was hyperbolic except for certain values, especially for the autonomous car density $\sigma = 0$. We also discovered that under a closed system, the coupled conservation laws, when its total density is expressed as $\rho + \sigma$, can be modeled as a linear one dimensional conservation equation.

Prior to our work, relatively little has been done in the experimental evaluation in autonomous vehicles. Most notably, the recent paper "Traffic Modeling—Phantom Traffic Jams and Traveling Jamitons" conducted by mathematician Morris R. Flynn [3]. Labeling shock waves as 'Jamitons', Flynn and his team investigated the buildup of cars in a road that arise in the absence of any obstacles. One key difference between our work and their work is that we have implemented human driven cars in the autonomous system as well. To the best of our knowledge, we are the first to implement such combinations of cars, and the first to code such results of the interactions between such vehicles.

The first and second sections are the introduction and background of our research. In Section 3, we introduce existing constitutive laws that model velocity $v(\rho)$ as a function of car density ρ. By describing the existing constitutive equations and terms, we are able to define a new coupled conservation law later on that accounts for both autonomous and human-driven vehicles. In Section 4, we explain the different numerical methods that mathematicians have developed over the years, each one arguably more efficient than the last. These numerical methods are implemented in coding programs such as Matlab or Octave to plot the linear density of cars and position to see the interaction of density curves over time steps. However, these are all numerical methods for a 1-D traffic system and cannot simulate a system with two kinds of vehicles. In Section 5, we propose a new system of partial differential conservation equations that model a traffic system with both autonomous and human vehicles, the densities of which is expressed as ρ and σ respectively. We proceed in our research by plugging in known constitutive laws for velocity $v(\rho)$. We also uses the Lax-Friedrichs scheme to simulate the solution. In Sections 6 and 7, we derive the Roe solver for a class of constitutive laws and analyze and draw conclusions from our simulations. Finally, in Section 8, we provide conclusions and discuss future work.

2 Background and Definitions

This section gives some background on how we will define and model the traffic flow of a system while defining some functions and variables that will be used in the paper later.

Definition 2.1. Density Frequently written as ρ, linear density is a variable in the modeled system on a road where the number of cars passes a specific unit length. In a changing system where time and position is non-constant, linear density can be written as a function $\rho(x, t)$.

Definition 2.2. Flux Flux is defined as the number of cars passing in a specific unit of time, or a time step. Similar to linear density, flux can be written as a function of two variables, ρ and v, as $J(\rho, v)$.

Although flux J can relate on other factors such as the position on the road and derivative of density ρ, in our research, we assume that flux is only affected by density ρ or $J = J(\rho)$. Because the units of flux is defined as $J = \frac{cars}{time}$, in small time steps and position intervals, the expression can be rewritten as $\frac{cars}{length} \cdot \frac{length}{time}$, which is $\rho \cdot v(\rho)$.

A macroscopic traffic model involving traffic flux, traffic density and velocity forms the basis of the fundamental diagram. By using this diagram, we can understand and predict the traffic laws we will implement in our macroscopic model.

2.1 The Traffic Model

When constructing a traffic model, many mathematicians begin with a conservation law, for which we can derive and generalize for a system of cars. One of the most important distinctions between reality and traffic modeling is when cars are not conserved within a system. There are multiple ways for cars to magically “reappear” and “disappear” within a system as shown in Figure 1 Thus, the reason why we use a conservation law is to assume the conservation of cars within our artificial traffic model.

Our definition of a conservation law, which states that cars can never be created nor destroyed in a fixed system, is similar to the First Law of Thermodynamics. One can represent this algebraically within a given interval Δx by taking the integral of density. The right hand side of the expression is the difference in density of the cars between the end and the beginning of the interval must be conserved. Mathematicians Lighthill, Whitman, and Richards refine this model as we see in section 2.2.

$$\frac{d}{dt}\int_{x_0}^{x_0+\Delta x} \rho(x,t)dx = J(x_0,t) - J(x_0+dx,t)$$

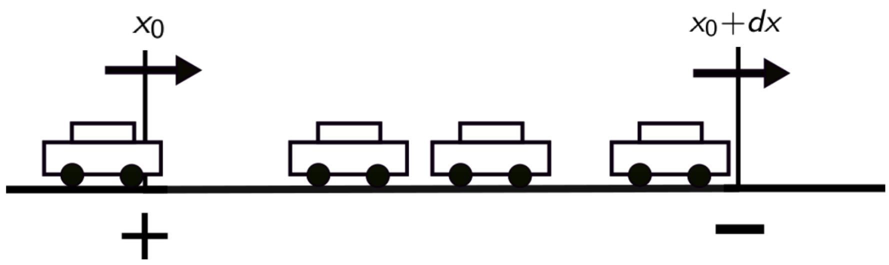

Figure 1: The integral definition of conservation of cars

2.2 Conservation Laws

The Lighthill-Whitman-Richards (LWR) Model is the fundamental macroscopic model, relating density and velocity at any position x and time t. This can be derived through noting the conservation of vehicles: number of cars in a given interval Δx at time t is the integral of the linear density throughout the interval, and also equal to the difference of the fluxes at the endpoints of the interval, where flux $J(\rho, v) = \rho(x,t) \cdot v(\rho)$. Thus, at any point x_0 at time t_0 we have:

$$2\Delta x(\rho(x_0, t_0+\Delta t), \rho(x_0, t_0-\Delta t)) = 2\Delta t(J(x_0-\Delta x, t_0) - J(x_0+\Delta x, t_0))$$

.

Using a first-order Taylor expansion and manipulating the terms in this equality simplifies it to a partial differential equation

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial J(\rho, v)}{\partial x} = 0, \quad x \in \mathbb{R}, \text{t} > 0$$

with the initial condition

$$\rho(x,0) = \rho_0(x), \quad x \in \mathbb{R}$$

This time-dependant system of partial differential equations is usually non-linear but take a particularly simple structure. The equation accurately describes any system with a conserved amount of particles. Given a initial distribution of density $\rho_0(x)$, solving the conservation law will allow us to know about the density on the road in a future time step. In one-dimension, the equations take the form of a hyperbolic conservation law. This conservation law can also be rewritten in the simplified form $\rho_t + J_x = 0$, where $\rho(x,t)$ is the linear density of cars on a road and $J(\rho, v)$ is the flux of cars.

In general, it is very difficult, if not impossible, to derive exact solutions to these equations, and hence the need to devise and study numerical methods for their approximate solution, as mathematicians have done so in the past few decades. Of course the same is true more generally for any nonlinear PDEs, and to some extent the general theory of numerical methods for nonlinear PDEs applies in particular to systems of conservation laws. However, there are several reasons for studying this particular class of equations on their own in some depth [4].

Many practical problems in science and engineering involve conserved quantities and lead to PDEs of this class. There are certain difficulties associated with solving these systems (e.g. shock formation) that are not seen elsewhere and must be dealt with carefully in developing numerical methods. Methods based on naive finite difference approximations may work well for smooth solutions but can give disastrous results when discontinuities are present, as we saw when our simulations 'blew up', or diverged very quickly. Although few exact solutions are known, a great deal is known about the mathematical structure of these equations and their solutions. This knowledge can be exploited to develop special methods that overcome some of the numerical difficulties encountered with a more naive approach, as we will see in our Roe solver [4].

2.3 Shock Waves

The basis of our research is on shock waves, a fundamental field of research in traffic flow. Mathematicians in Japan conducted an experiment where they placed a number of cars around a circular fixed road and instructed the human drivers to drive at constant speed. Despite such instructions, distances between the cars eventually started to vary, causing a buildup of cars and forming a traffic jam, or as mathematicians would describe it, a shock wave, that traveled backwards. This phenomenon has also been observed in a regular road system as well.

Definition 2.3. Shock wave A shock wave is a discontinuity of flow or density, and has the physical implication that cars change speeds abruptly without time to accelerate or decelerate.

This unnatural behaviour of cars can be eliminated by considering high order continuum models. However, such equations are extremely difficult to model due to its complexity.

3 Constitutive Equations

Constitutive equations are functions that relate two physical quantities to each other under the assumption that they are not dependant on other external variables and factors. They are commonly used to model non-constant phenomenon such as the deformation of solids and fluids, electromagnetism, and friction. In our case, we examine existing constitutive equations for traffic modeling developed by mathematicians.

Constitutive equations for human-driven vehicles have long been developed by mathematicians for decades, describing the interactions between human-driven vehicles only. Evolving from a single-lane system, variations of the original constitutive traffic laws have been developed, such as a multiple lane system or an addition of ramps such that cars can enter or leave the system, where the original constitutive equations fall apart due to flux not being conserved.

The proposed constitutive equations are actually similar to the that of human-driven vehicles in that the general curvature of the velocity and density function is preserved. However, there are new properties of these new laws. Comparing to human drivers, the autonomous vehicles have an almost instantaneous reaction time and have communication between vehicles. These features allow them to drive at a higher speed than human drivers when the density is not too high. When density increases to certain degree, to avoid collision, the autonomous vehicles, like human drivers will lower their speed. Ergo, certain parameters can be controlled variables, ones that we can manipulate in our code when plugging in the conservation laws in order to model the behaviour of these autonomous vehicles.

Although there exist many functions that model $v(\rho)$ as a function of linear density ρ, such as the arctangent function and the Gauss error (erf) function with a more natural and realistic curvature, our research only focuses on the application of three of the simplest but most commonly-used constitutive laws.

Most constitutive laws for $v(\rho)$ model the nature of a maximum velocity when linear density $\rho = 0$, and of velocity $v = 0$ when density is maximized due to the nature of traffic. Theoretically, in a clear road where linear density would be 0, a car can cruise at maximum velocity v_m. However, in a congested road where linear density is maximized, the velocity v of a car should be approximately, or exactly equal to, 0.

3.1 Linear Piecewise Function

Although other constitutive laws model traffic flow more accurately, one of the simplest general functions modeled by our data is the linear piecewise function: a density function where initial velocity is constant before the vehicle approaches a traffic jam, and its velocity dramatically decreases linearly, as described below and illustrated in Figure 2.

$$v(\rho) = \begin{cases} v_m & (\rho \leq \rho_c) \\ -\frac{v_m}{\rho_m - \rho_c}\rho + \frac{v_m \rho_m}{\rho_m - \rho_c} & (\rho_c < \rho \leq \rho_m) \end{cases}$$

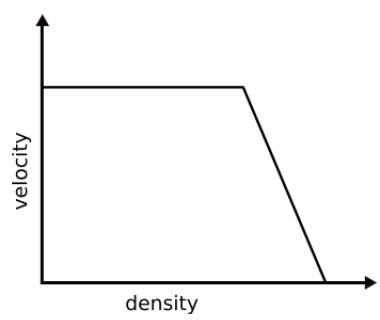

Figure 2: linear piecewise function estimating a general density curve

3.2 Burger's Equation

A fundamental partial differential equation occurring in various areas of applied math, Burger's Equation was derived by mathematician Johannes Martinus Burgers. The viscid form of the equation is written

$$u_t + uu_x = \epsilon u_{xx}$$

where $\epsilon > 0$ is the viscosity or diffusion constant [5]. This is the simplest partial differential equation combining both propagation and diffusion. However, because our research examines the conservative nature of a system, we only work with the inviscid form of the equation, which can we rewritten as

$$\rho_t + \rho\rho_x = 0 \quad \text{where} \quad J(\rho, v) = \frac{\rho^2}{2}$$

with no diffusive terms. However, one pitfall to the model is that it does not consider human factors and individual cars, and that it is limited to a single lane system.

3.3 Greenshields Model

In 1935, mathematician Bruce D. Greenshields made the assumption that, under uninterrupted flow conditions, speed and density are linearly related [1]. While Greenshields' model is not perfect, it is fairly accurate and relatively simple. This relationship can be expressed mathematically and graphically, as shown below in Figure 3 [1]. Greenshields defined his constitutive equation as

$$v(\rho) = v_m\big(1 - \frac{\rho(x,t)}{\rho_m}\big)$$

. Therefore, the flux equation, $J(\rho, v)$, can be expressed as

$$J(\rho, v) = v_m \cdot \rho(x,t)\big(1 - \frac{\rho(x,t)}{\rho_m}\big)$$

In Figure 3.3, notice that when linear density, $\rho(x,t)$, approaches 0, the cars are able to reach maximum velocity v_m and when $\rho(x,t)$ approaches maximum density ρ_m, the cars slow down to zero velocity. However, like the Burger's Equation, one pitfall to the model is that it does not consider human factors and individual cars, and that it is limited to a single lane system.

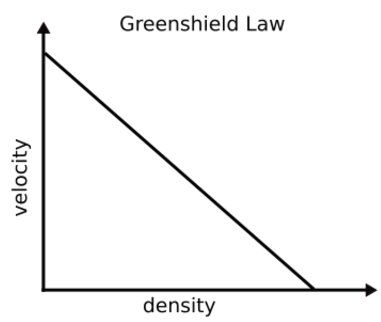

Figure 3: Greenshields model of velocity vs density is linear

Note that the expression $\frac{\rho(x,t)}{\rho_m}$ could be raised to an integer power to manipulate the curvature of the velocity vs density graph, and thus changing our density curves in our simulations. Thus, another constitutive law arises as shown below.

$$J(\rho, v) = v_m \cdot \rho(x,t)\big(1 - \big(\frac{\rho(x,t)}{\rho_m}\big)^n\big) \quad \text{where } n \in \mathbb{N}$$

4 Methods of Solving the Conservation Law

This section will explain the methods used to solve the conservation law. When working with partial differential equations, there are many different ways of solving such characteristics. Over the years, mathematicians have derived many different ways of solving such equations, graphing the density curves over time in a simulation with today's technology. One such method is the numerical method; by plugging in numerical schemes in coding programs such as Matlab or Octave, one can plot densities over a road in small time steps. Albeit occasional bugs and density curves 'blowing up', mathematicians are able to plot such conservation equations when unable to solve them analytically to a certain degree of accuracy.

Note the change in notation. Subscripts $\overline{\rho}_m$ are not defined as the partial derivative of linear density in terms of m, but rather the density of the noted position m in our system, while the superscript k is defined as the density of that moment in time k at that particular position m. The overline is defined as the average density at that position m.

4.1 Method of Characteristics

The method of characteristics is a technique for solving PDEs. Although most commonly used in solving first-order equations, the method can be used to solve any general hyperbolic partial differential equation.

When working with the method of characteristics, we try to find the solution of the equation on certain curves called characteristics. In the case of the conservation law, the curves are infinitesimal lines described by

$$x = x_0 + J'(\rho)t$$

. Because $\frac{dx}{dt} = J'(\rho)$ for these density curves, the value of ρ does not change. Thus, the characteristics carry the initial density to a future time step. To obtain the density at (x, t), all we need to do is identify the characteristics that pass through that point and find the density it carries, and manually calculate the density during the next time step at that specific position in the system. An example of how characteristics correspond to the density curve is shown in Figure 4.

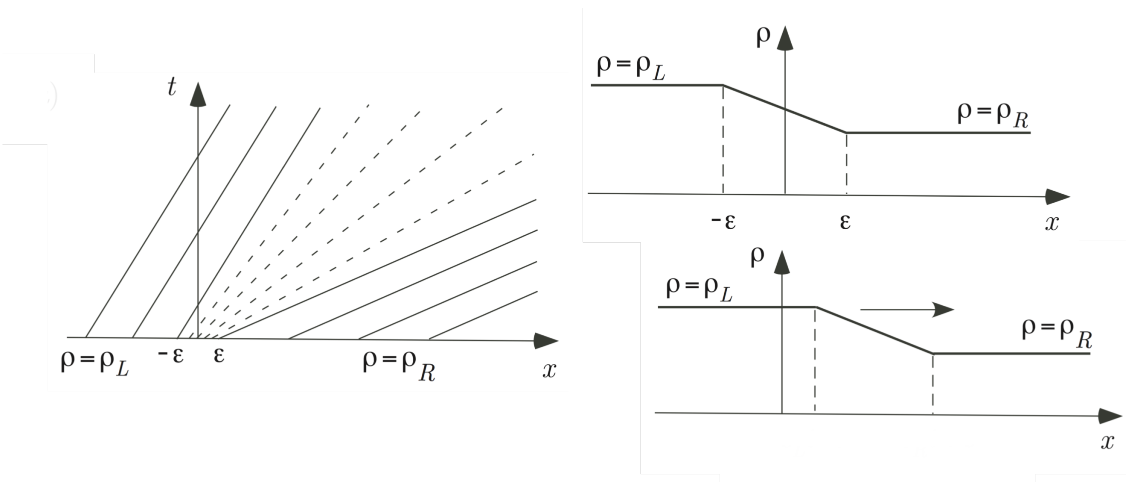

Figure 4: Correspondence between Density Curves and Characteristics [3]

4.2 Finite Volume Method

When there are complicated constitutive equations and shock waves occur, it may be difficult to trace the characteristics and find the solution analytically. The Finite Volume Method is a common numerical method to approximate the solution. It divides the road into cells and calculate how the density inside the cell change by considering the flux in and out of the cell. The left hand side is the rate of change in linear density at a position x, while the right hand side is the approximate change in flux at position x.

$$\frac{h}{k}(U_i^{n+1} - U_i^n) = f(U_{i-\frac{1}{2}}^n) - f(U_{i+\frac{1}{2}}^n)$$

One can substitute the constants $\overline{\rho} = U$ and $J = f(U)$ and rewrite and achieve the numerical recursion in terms of linear density ρ and flux J.

$$\frac{\Delta x}{\Delta t}(\overline{\rho}_x^{t+1} - \overline{\rho}_x^t) = J_{x-\frac{1}{2}}^t - J_{x+\frac{1}{2}}^t$$

Thus, simplifying with algebra, we can rewrite this to solve for unknown future density in a recursive form:

$$\overline{\rho}_x^{t+1} = \overline{\rho}_x^t + \frac{\Delta t}{\Delta x}(J_{x-\frac{1}{2}}^t - J_{x+\frac{1}{2}}^t).$$

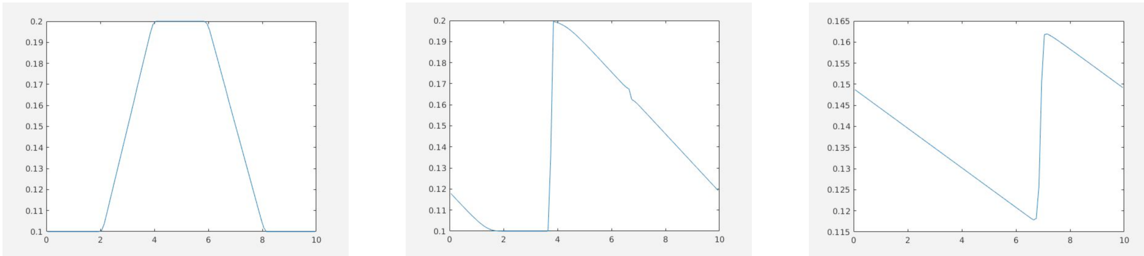

Figure 5: Solution for a 1-D System over time with the Upwind Method

4.2.1 The Upwind Method

The upwind method is a kind of finite volume method that we use for traffic systems that only have one kind of car. The upwind method calculates the intercell flux based on the direction of the characteristics. If the left travels to the right, we take the density at the immediate left to calculate future densities, and perform similar calculations with the right, if the characteristics are traveling to the left. Based on the upwind method, the flux between cell x and $x+1$ at a certain time t is given by the following equation.

$$J(\overline{\rho}_x, \overline{\rho}_{x+1}) = \frac{1}{2}\big(J(\overline{\rho}_x) + J(\overline{\rho}_{x+1}) - a(\overline{\rho}_{x+1} - \overline{\rho}_x)\big)$$

In this equation, $a = |J'(\overline{\rho}_x)|$ if $\overline{\rho}_x = \overline{\rho}_{x+1}$ and $a = |\frac{J(\overline{\rho}_x)-J(\overline{\rho}_{x+1})}{\overline{\rho}_x-\overline{\rho}_{x+1}}|$ if $\overline{\rho}_x \neq \overline{\rho}_{x+1}$. Notice that once we get the intercell fluxes, we can calculate the densities of the next time step using the differences of each adjacent fluxes, as shown in Figure 5. The upwind method is fairly accurate and is useful when we are trying to observe the behavior in a traffic system with only one kind of car. The core structure of the upwind method code is shown in Appendix A.

5 Coupled Equations

The aim of our research is to provide the simplest yet accurate model for modeling the relationship between human drivers and autonomous cars. Here, we consider a traffic system that contains both autonomous vehicles and human-driven vehicles. We use ρ and J_1 to represent the density and flux of human-driven cars and σ and J_2 to represent the density and flux of autonomous vehicles. The following system of conservation equations is the general form that describe a mixed traffic system.

$$\begin{cases} \rho_t + \Big(J_1(\rho,\sigma)\Big)_x = 0 \\ \sigma_t + \Big(J_2(\rho,\sigma)\Big)_x = 0 \end{cases}$$

However, we assume the traffic system in which the velocity of the cars depends on the total density rather than the respective density of human and autonomous vehicles. We then have $v_1(\rho,\sigma) = v_1(\rho+\sigma)$ and $v_2(\rho,\sigma) = v_2(\rho+\sigma)$. This means that each car cannot distinguish between autonomous and human vehicles and treat them indiscriminately. The flux of human vehicles is thus $J_1(x,t) = v_1(\rho+\sigma)\cdot\rho$ and the flux of autonomous vehicles is $J_2(x,t) = v_2(\rho+\sigma)\cdot\sigma$. We also assume that the velocity decreases as the density increases. Below is the set of equations that we are trying to solve.

$$\begin{cases} \rho_t + \Big(v_1(\rho+\sigma)\cdot\rho\Big)_x = 0 \\ \sigma_t + \Big(v_2(\rho+\sigma)\cdot\sigma\Big)_x = 0 \end{cases}$$

From (10), we can evaluate the partial derivatives and rewrite the expression. Notice that v_i' here equals $\frac{\partial v_i}{\partial \rho}$ and $\frac{\partial v_i}{\partial \sigma}$.

$$\begin{cases} \rho_t + v_1'(\rho+\sigma)\cdot(\rho_x+\sigma_x)\rho + v_1(\rho+\sigma)\cdot\rho_x = 0 \\ \sigma_t + v_2'(\rho+\sigma)\cdot(\rho_x+\sigma_x)\sigma + v_2(\rho+\sigma)\cdot\sigma_x = 0 \end{cases}$$

We can rewrite this new expression as a 2 $\times$ 2 matrix due to the symmetry between ρ and σ.

$$\begin{pmatrix}\rho\\ \sigma\end{pmatrix}_t + \begin{pmatrix} v_1'\rho + v_1 & v_1'\rho \\ v_2'\sigma & v_2'\sigma + v_2\end{pmatrix}\begin{pmatrix}\rho\\ \sigma\end{pmatrix}_x = 0$$

5.1 Solving for the Characteristics of the Conservation Laws

By working with matrices and their eigenvalues, we can analyze the characteristics of the coupled equation and determine whether that system is hyperbolic. To do so, we first must consider the velocity functions that can model such behaviour. Newton's Method, similar to the method of characteristics, is described as

$$\begin{pmatrix} f_1\\ f_2\end{pmatrix} \approx \begin{pmatrix} f_{1_0}\\ f_{2_0}\end{pmatrix} + \nabla f(u_0)\cdot(u-u_0)\,.$$

Newton's Method reveals that for a conservation law $u_t + f(u)_x = 0$, we can rewrite $f(u)$ as a matrix. Notice that in (13), $\nabla f(u_0) \approx f'(u_0) = \boldsymbol{J}$ where $\boldsymbol{J}$ is the flux scalar matrix of the vector $\langle \rho, \sigma\rangle$.

$$\nabla f(u_0) = f'(u_0) = \boldsymbol{J} = \begin{pmatrix} \frac{\partial J_1}{\partial \rho} & \frac{\partial J_1}{\partial \sigma} \\ \frac{\partial J_2}{\partial \rho} & \frac{\partial J_2}{\partial \sigma}\end{pmatrix}$$

If the eigenvalues of our 2 x 2 matrix are distinct and real, the matrix itself will be diagonalizable and the system will be strictly hyperbolic and symmetric. Then, we solve for the matrix's eigenvalues to determine the characteristics of the matrix.

$$\det\begin{vmatrix} \frac{\partial J_1}{\partial \rho} - \lambda & \frac{\partial J_1}{\partial \sigma} \\ \frac{\partial J_2}{\partial \rho} & \frac{\partial J_2}{\partial \sigma} - \lambda\end{vmatrix} = 0$$

$$\Big(\frac{\partial J_2}{\partial \sigma} - \lambda\Big)\Big(\frac{\partial J_1}{\partial \rho} - \lambda\Big) - \frac{\partial J_1}{\partial \sigma}\frac{\partial J_2}{\partial \rho} = 0$$

$$\lambda^2 - \Big(\frac{\partial J_2}{\partial \sigma} + \frac{\partial J_1}{\partial \rho}\Big)\lambda + \Big(\frac{\partial J_2}{\partial \sigma}\frac{\partial J_1}{\partial \rho} - \frac{\partial J_1}{\partial \sigma}\frac{\partial J_2}{\partial \rho}\Big) = 0$$

From here we obtain a quadratic for which we can solve and find various properties of the two eigenvalues, λ_1 and λ_2. We then take the discriminant of the quadratic,

$$\Delta : \Big(\frac{\partial J_2}{\partial \sigma} + \frac{\partial J_1}{\partial \rho}\Big)^2 - 4\Big(\frac{\partial J_2}{\partial \sigma}\frac{\partial J_1}{\partial \rho} - \frac{\partial J_1}{\partial \sigma}\frac{\partial J_2}{\partial \rho}\Big) \geq 0$$

. Notice that we can factor and complete the square of the expression:

$$\Delta : \Big(\frac{\partial J_2}{\partial \sigma} - \frac{\partial J_1}{\partial \rho}\Big)^2 + 4\Big(\frac{\partial J_2}{\partial \sigma}\frac{\partial J_1}{\partial \rho} - \frac{\partial J_1}{\partial \sigma}\frac{\partial J_2}{\partial \rho}\Big) \geq 0\,.$$

We then plug in the components of the matrix from (12), getting:

$$\Big((v_1'\rho + v_1) - (v_2'\sigma + v_2)\Big)^2 + 4v_1'v_2'\rho\sigma \geq 0\,.$$

Since the squared expression is always at least 0 and $4v_1'v_2'\rho\sigma$ is always non-negative, the discriminant will be at least 0 and there will be at least one eigenvalue. In most cases, we will have two distinct eigenvalues which indicates that the system is strictly hyperbolic. However, there are specific points on the ρ - σ plane that have $\Delta = 0$. In the common case when $v_i' < 0$, there can at most exist two of those points: one that has $\rho = 0$ and one that has $\sigma = 0$. Because the system only has the discriminant $\Delta = 0$ for two specific pairs of ρ and σ, we can claim for the system to be generally hyperbolic.

5.2 Lax-Friedrichs Method

The Lax-Friedrichs Method is a crude numerical method that we use to approximate the solution of the coupled conservation equation. The method derives from the linear hyperbolic partial differential equation for $u(x,t)$ in the form $u_t + uu_x = 0$. In the expression, like the upwind and finite volume method, the subscripts and the superscripts are variables representing the position and grid intervals. In our case, we take the average density of the interval and calculate and subtract the two fluxes of the averaged positions before and after. Then the Lax-Friedrichs method for solving the partial differential equation is given by [4]:

$$\frac{U_i^{n+1} - \frac{1}{2}(U_{i+1}^n + U_{i-1}^n)}{k} + a\frac{f(U_{i+1}^n) - f(U_{i-1}^n)}{2h} = 0\,.$$

Here, U is the density vector, k is the time between each time step, and h is the cell length. In the case of the scalar linear problem we are working with in the form of Burger's equation, we can set $a = 1$. Then, simplifying with algebra, we can rewrite this to solve for the unknown future value U_i^{n+1} as shown below.

$$U_i^{n+1} = \frac{1}{2}(U_{i+1}^n + U_{i-1}^n) - \frac{k}{2h}(f(U_{j+1}^n) - f(U_{j-1}^n))$$

By substitution of our variables linear density $\begin{pmatrix}\overline{\rho}\\ \overline{\sigma}\end{pmatrix} = U$ and $\begin{pmatrix}J_1\\ J_2\end{pmatrix} = f(U)$, this method can be written in the conservation form for our traffic model

$$\overline{\rho}_x^{t+1} = \overline{\rho}_x^t - \frac{\Delta t}{\Delta x}[J_1(x, x+1) - J_1(x-1, x)]$$

$$\overline{\sigma}_x^{t+1} = \overline{\sigma}_x^t - \frac{\Delta t}{\Delta x}[J_2(x, x+1) - J_2(x-1, x)]$$

where $J_i(x, x+1)$ is the inter-cell flux between cell x and cell $x+1$ and it is given by

$$J_1(x, x+1) = \frac{\Delta t}{2\Delta x}(\overline{\rho}_x - \overline{\rho}_{x+1}) + \frac{1}{2}(J_{1x} + J_{1x+1})$$
$$J_2(x, x+1) = \frac{\Delta t}{2\Delta x}(\overline{\sigma}_x - \overline{\sigma}_{x+1}) + \frac{1}{2}(J_{2x} + J_{2x+1})$$

However, a pitfall of the Lax-Friedrichs Method is that the numerical method implements diffusive terms, smoothing out the density curves when they should not be, leading to inaccurate solutions. Figure 5 is a screen-shot of our coupled equations interacting with each other in a specific. In our simulation, we plotted a Greenshield's Model and a high-degree polynomial of Greenshield's Model with the same maximum velocity, where the density curves would change over time as the time-steps increased. Below are the two constitutive laws that we used to produce the solution in Figure 6.

$$\begin{cases} J_1 = v_m\left(1 - \frac{\rho+\sigma}{p_m}\right)\cdot\rho \\ J_2 = v_m\left(1 - (\frac{\rho+\sigma}{p_m})^{20}\right)\cdot\sigma \end{cases}$$

In Figure 6, the y-axis is defined as the density ρ while the x-axis is also defined as the specific position along the road. J_1 is the blue density curve of human-driven cars, while J_2 is the red density curve of autonomous cars. The core structure of the Lax-Friedrichs code is shown in Appendix B.

6 Roe Solver for the Coupled Equation

Derived by mathematician Phil Roe, the Roe solver involves approximately solving the Riemann problem between every adjacent cells by considering it in a system with linear flux. We can rewrite the coupled equation as $\vec{u}_t + J\vec{u}_x = 0$ where $\vec{u}$ is the vector $\langle\rho, \sigma\rangle$ and J is the Jacobian matrix. For every two adjacent cells with u_L

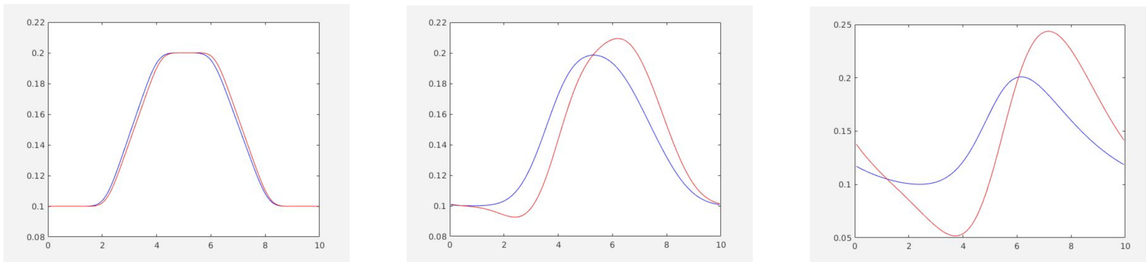

Figure 6: Solution for a 2-D System over time with the Lax-Friedrichs Method

and u_R, the Roe solver approximates the inter-cell flux by solving the linear system of $u_t + \hat{R}(u_L, u_R)u_x = 0$ where the Roe matrix $\hat{R}$ is a constant matrix and $\hat{R}u$ approximates the actual flux $f(u)$. In his paper, Roe indicated the three conditions that the Roe matrix must follow [4]:

i $\hat{R}(u_L - u_R) = f(u_L) - f(u_R)$

ii $\hat{R}$ is diagonalizable with real eigenvalues

iii $\lim_{u_L, u_R \to u} \hat{R}(u_L, u_R) = f'(u)$

6.1 Derivation of Roe Matrix for Polynomial Flux Laws

Due to these restrictions on the Roe matrix, for each set of flux rules, we must to analytically come up with a Roe matrix that must satisfy the three conditions. This section presents the method we discovered to derive the Roe matrix for 2-D polynomial flux laws. To satisfy the first condition of the Roe matrix, we must have:

$$J_i(\rho_L, \sigma_L) - J_i(\rho_R, \sigma_R) = a_i(\rho_L - \rho_R) + b_i(\sigma_L - \sigma_R)\,.$$

Here, J_i is the polynomial flux function and a_i and b_i are some expressions of ρ_L, ρ_R, σ_L, and σ_R. The polynomial function J_i can be expressed as the sum of terms in the form of $k_{mn}\rho^m\sigma^n$, where k_{mn} is the coefficient. The difference for the term is $k_{mn}(\rho_L^m\sigma_L^n - \rho_R^m\sigma_R^n)$, which we can express as the following:

$$\begin{aligned} k_{mn}(\rho_L^m\sigma_L^n - \rho_R^m\sigma_R^n) &= \frac{k_{mn}}{2}((\sigma_L^n + \sigma_R^n)(\rho_L^m - \rho_R^m) + (\rho_L^m + \rho_R^m)(\sigma_L^n - \sigma_R^n)) \\ &= \frac{k_{mn}}{2}((\sigma_L^n + \sigma_R^n)(\sum_{j=0}^{m-1} \rho_L^j \rho_R^{m-1-j})(\rho_L - \rho_R) + \\ &\quad (\rho_L^m + \rho_R^m)(\sum_{j=0}^{n-1} \sigma_L^j \sigma_R^{n-1-j})(\sigma_L - \sigma_R)) \end{aligned}$$

Let $a_{mn} = \frac{k_{mn}}{2}(\sigma_L^n + \sigma_R^n)(\sum_{j=0}^{m-1} \rho_L^j \rho_R^{m-1-j})$ and $b_{mn} = \frac{k_{mn}}{2}(\rho_L^m + \rho_R^m)(\sum_{j=0}^{n-1} \sigma_L^j \sigma_R^{n-1-j})$. Then, a_{mn} and b_{mn} are the expressions that allow us to have the difference for the term $k_{mn}\rho^m\sigma^n$ in the form of $a_{mn}(\rho_L - \rho_R) + b_{mn}(\sigma_L - \sigma_R)$. Adding the expressions for each term together, we obtain the expressions needed for polynomial flux function J_i :

$$\begin{aligned} J_i(\rho_L, \sigma_L) - J_i(\rho_R, \sigma_R) &= (\sum a_{mn})(\rho_L - \rho_R) + (\sum b_{mn})(\sigma_L - \sigma_R) \\ &= a_i(\rho_L - \rho_R) + b_i(\sigma_L - \sigma_R) \\ &. \end{aligned}$$

We can calculate a_1 and b_1 for J_1 and a_2 and b_2 for J_2. Then, we obtain:

$$J(u_L) - J(u_R) = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \rho_L - \rho_R \\ \sigma_L - \sigma_R \end{pmatrix}$$

Ergo, the matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ satisfies the first condition of the Roe matrix. For the specific term $k_{mn}\rho^m\sigma^n$, its partial derivatives with respect to ρ and σ are $mk_{mn}\rho^{m-1}\sigma^n$ and $nk_{mn}\rho^m\sigma^{n-1}$. We then have:

$$\begin{cases} \lim_{u_L,u_R\to u} \frac{k_{mn}}{2}(\sigma_L^n+\sigma_R^n)(\sum_{j=0}^{m-1}\rho_L^j\rho_R^{m-1-j}) = & mk_{mn}\rho^{m-1}\sigma^n \\ \lim_{u_L,u_R\to u} \frac{k_{mn}}{2}(\rho_L^m+\rho_R^m)(\sum_{j=0}^{n-1}\sigma_L^j\sigma_R^{n-1-j}) = & nk_{mn}\rho^m\sigma^{n-1} \end{cases}$$

This shows that for each term of J_i, the a_{mn} and b_{mn} equal to the partial derivatives of ρ and σ, respectively. Therefore, the matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ satisfies the third condition that it equals to the Jacobian as $u_L, u_R \to u$.

So far, the matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ satisfy conditions i and iii. It is likely that it also satisfy the second condition of being diagonalizable with real eigenvalues. This condition requires that the discriminant $(a_1-b_2)^2-4b_1a_2>0$. As discussed before, because of special cases when $\rho=0$ or $\sigma=0$, we can at best have $(a_1-b_2)^2-4b_1a_2\geq 0$. Notice that a_{mn} and b_{mn} are formed by using some averaging between ρ_L and ρ_R and between σ_L and σ_R as values for the derivatives. For example, $\frac{k_{mn}}{2}(\sigma_L^n+\sigma_R^n)(\sum_{j=0}^{m-1}\rho_L^j\rho_R^{m-1-j})$ is formed by plugging in $\overline{\rho}=(\frac{\sum_{j=0}^{m-1}\rho_L^j\rho_R^{m-1-j}}{m})^{\frac{1}{m-1}}$ and $\overline{\sigma}=(\frac{\sigma_L^n+\sigma_R^n}{2})^{\frac{1}{n}}$ in $\frac{\partial J_i}{\partial\rho}$. Therefore, if k_{mn} is positive, we have:

$$mk_{mn}\rho_{\min}^{m-1}\sigma_{\min}^n \leq a_{mn} \leq mk_{mn}\rho_{\max}^{m-1}\sigma_{\max}^n\,.$$

Here, $\rho_{\min}=\min(\rho_L,\rho_R)$ and so on. If k_{mn}, the coefficients in J_i, are all positive or negative, we have $(a_1-b_2)^2-4b_1a_2\geq 0$, where equality can only be achieved when $\rho_L=\rho_R=0$ or $\sigma_L=\sigma_R=0$. Thus, in such a case, the second condition of being diagonalizable with real eigenvalues is generally satisfied. However, it is possible that $(a_1-b_2)^2-4b_1a_2<0$ when some coefficients are positive and others are negative. Therefore, we should check whether $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ satisfies the second condition. In most cases, it meets the second condition and is in fact a Roe matrix.

Below is an example of a Roe solver with a polynomial variation of Greenshields' Law derived from (5). The flux rule we have determined for this Roe solver is

$$\begin{cases} v_1 = v_{m_1}\left(1-\frac{\rho+\sigma}{\rho_m}\right) \\ v_2 = v_{m_2}\left(1-(\frac{\rho+\sigma}{\rho_m})^2\right) \end{cases}$$

where J_1 is the flux of human vehicles, J_2 is the flux of autonomous vehicles, and ρ_m is the maximum density that causes cars to stop. Below is the Roe matrix for this set of flux rules:

$$\boldsymbol{R} = \begin{pmatrix} 1-\frac{2\rho_L+2\rho_R+\sigma_L+\sigma_R}{2\rho_m} & -\frac{\rho_L+\rho_R}{2\rho_m} \\ -\frac{(\rho_L+\rho_R)(\sigma_L+\sigma_R)+2\sigma_L^2+2\sigma_R^2}{2{\rho_m}^2} & 1-\frac{\rho_L^2+\rho_R^2+2(\rho_L+\rho_R)(\sigma_L+\sigma_R)+2\sigma_L^2+2\sigma_R^2+2\sigma_L\cdot\sigma_R}{2{\rho_m}^2} \end{pmatrix}$$

As mentioned before, condition (i) and (iii) are guaranteed to be met. After checking, we see that condition (ii) is met only when neither $\rho_L=\rho_R=0$ nor $\sigma_L=\sigma_R=0$ is true. Thus, our matrix satisfies the conditions for a Roe matrix.

6.2 The Roe Solver

After deriving the Roe matrix $\hat{R}$ for the two flux laws, we can then code the Roe solver. As a finite volume method, the Roe solver requires the calculation of intercell fluxes for each adjacent cell. As explained at the end of section 6.1, for a pair of adjacent cells in which ρ_L, ρ_R, σ_L and $\sigma_R \neq 0$, all conditions of the Roe matrix are satisfied. In this case, we calculate the intercell flux by considering the Riemann problem in the linear system $U_t + \hat{R}U_x = 0$, where $\hat{R}$ is the roe matrix derived by plugging in $\rho_L, \rho_R, \sigma_L, \sigma_R$ value at that pair of cells. To calculate the intercell flux, we perform the eigendecomposition on $\hat{R}$, resulting in eigenvalues λ_1, λ_2 and eigenvectors r_1, r_2. The Roe matrix $\hat{R}$ then can be rewritten as QDQ^{-1} where D is the diagonal matrix formed by the eigenvalues and Q is a matrix that has r_1 and r_2 as columns. Then, the decoupled system can be written in the following form.

$$\begin{cases} r_{1t} + \lambda_1 r_{1x} = 0 \\ r_{2t} + \lambda_2 r_{2x} = 0 \end{cases}$$

The decoupled system simply describes two discontinuities, one in r_1 and one in r_2, travelling in the speed of λ_1 and λ_2, respectively. In this system, the flux vector F is simply $\begin{pmatrix} \lambda_1 r_1 \\ \lambda_2 r_2 \end{pmatrix}$. Then, the intercell fluxes of the $U_t + \hat{R}U_x = 0$ are given by $J = QF$, which is then used to calculate the densities of the next time step.

However, for a pair of adjacent cells that have $\rho_L = \rho_R = 0$ or $\sigma_L = \sigma_R = 0$, the second condition of the Roe matrix might not hold. In this case, because there is only one density left and we are left with a 1-D Riemann problem, we simply use the upwind method to obtain the flux. The core structure of our Roe solver code is shown in Appendix C.

7 Analysis of the Numerical Solution from the Roe Solver

When compared to the Lax-Friedrichs method, the Roe solver has a much higher resolution. While discontinuities in densities get smoothed out by the severe numerical diffusion in Lax-Friedrichs method, they are preserved in the Roe solver, allowing us to see how they transform overtime in the system. This section provides some qualitative observations on the traffic system that uses the constitutive laws we defined in (25) as $v_1 = v_{m_1}\big(1 - \frac{\rho+\sigma}{\rho_m}\big)$ and $v_2 = v_{m_2}\big(1 - (\frac{\rho+\sigma}{\rho_m})^2\big)$.

Through observing numerical solutions, we analyze that in a traffic system with only one kind of vehicle, the solution tends to be a rare-fraction fan followed by a discontinuous jump as shown in Figure 7. By using an Ansatz on the system, we can prove that the discontinuity in densities and the slope of the rare-fraction fan decreases until everything smooth out.

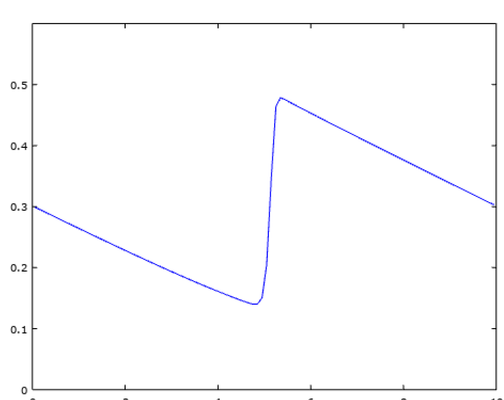

Figure 7: Solution for a 1-D System

Similar behavior occurs in a 2-D traffic system with two kinds of cars. Figure 8 shows a typical solution for the 2-D system. In the figure, the red curve describes the density of human-driven cars, the blue curve describes the density of autonomous cars, and the green curve describes the average density. As shown in the figure, the average density behaves similar to that of the 1-D solution curve in the way it consists of a rare-fraction fan followed by a shock. The discontinuity in density, or

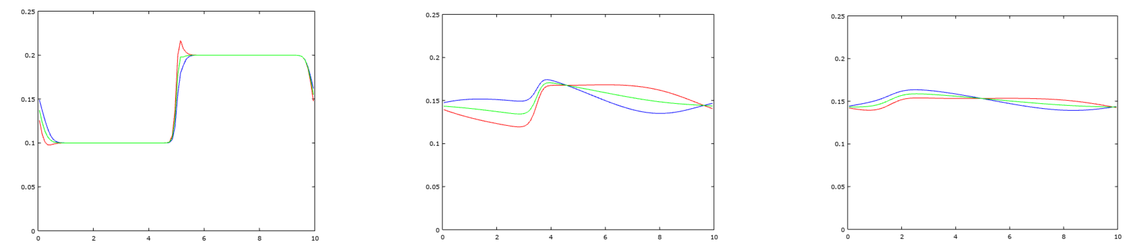

Figure 8: Solution for a 2-D System over time

the shock, of the green curve also decays as the 1-D solution curve does. The red and blue curves behaves differently from the green curve as they seems to oscillate around the green curve in periodic waves, which also gradually diminishes. However, the behavior described above is only an observation and needs to be proven by an Ansatz.

8 Conclusion and Future Investigations

In our work, we investigated the interaction between autonomous vehicles and human-driven vehicles, and discovered in our simulations that in a conserved system of density $\rho + \sigma$, the shock waves seem to be diffusing over time. Analyzing the graph, we observed that over time, the amplitude of the density curves gradually decreased. Although we have yet to prove this, we also hypothesized that the density curve of $\rho + \sigma$ was actually similar to that of a linear Greenshields' density curve.

One way to approach the analysis of our Roe solver is to use Stability Theory. Stability Theory addresses the stability of solutions and differential equations under small perturbations of initial conditions. This would allow people to fully prove the diffusion of the density curve of $\rho + \sigma$ as we had observed in our Octave code simulations.

It would be useful to create different Roe solvers with different constitutive laws, analyze the density curves, and compare how the properties of our Greenshields Roe solver are reflected in other constitutive laws. During those comparisons, one might come up with specific parameters to describe the efficiency and safety of a traffic systems. For example, one can measure the time it takes for shock waves to dissolve. This will then allow future mathematicians to propose what kind of traffic system functions the best.

9 Acknowledgements

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10 Appendix

Below, we compiled some of our code from Octave as a skeletal structure.

10.1 Appendix A: The Upwind Method

The core structure of the Upwind method Octave code we used is shown below. The car density is written as u, and i, j and F represents time steps, cells, and the intercell flux, respectively.

```
for i = 2:M
    for j = 1:(N-1)
        if u(j,i-1) == u(j+1,i-1)
            a = abs(Jprime(u(j,i-1)));
        else
            a=abs((J(u(j+1,i-1))-J(u(j,i-1)))/(u(j+1,i-1)-u(j,i-1)));
        end
        F(j+1) = 1/2*(J(u(j+1,i-1))+J(u(j,i-1))
            -1/2*a*(u(j+1,i-1)-u(j,i-1));
    end
    for j =1:N
        u(j,i) = u(j,i-1) + dt/dx*(F(j)-F(j+1));
    end
end
```

10.2 Appendix B: Lax Friedrichs

The core structure of the Lax-Friedrichs code we created is shown below, in which u is the density of human-driven cars, v is the density of autonomous cars, $J1$ is the constitutive law for human-driven cars, $J2$ is the constitutive law for autonomous cars, $F1$ is the intercell flux for human-driven cars, and $F2$ is the intercell flux of autonomous cars.

```
for i = 2:M
    for j = 1:(N-1)
        F1(j+1) = 1/2*(J1(u(j+1,i-1),v(j+1,i-1))
        +J1(u(j,i-1),v(j,i-1)))
        +1/2*(dx/dt)*(-u(j+1,i-1)+u(j,i-1));
        F2(j+1) = 1/2*(J2(u(j+1,i-1),v(j+1,i-1))
        +J2(u(j,i-1),v(j,i-1)))
        +1/2*(dx/dt)*(-v(j+1,i-1)+v(j,i-1));
    end
    for j =1:N
        u(j,i) = u(j,i-1) + dt/dx*(F1(j)-F1(j+1));
        v(j,i) = v(j,i-1) + dt/dx*(F2(j)-F2(j+1));
    end
end
```

10.3 Appendix C: Roe Solver

```
for i = 2:M
    for j = 1:(N-1)
        if (rho(j,i-1)>0) && (rho(j+1,i-1)>0)
        && (sig(j,i-1)>0) && (sig(j+1,i-1)>0)
            rhoL=rho(j,i-1);
            sigL=sig(j,i-1);
            rhoR=rho(j+1,i-1);
            sigR=sig(j+1,i-1);
            R=[1-(2*rhoL+2*rhoR+sigL+sigR)/(2*a), -(rhoL+rhoR)/(2*a);
            -((rhoL+rhoR)*(sigL+sigR)+2*sigL^2+2*sigR^2)/(2*a^2),
            1-(rhoL^2+rhoR^2+2*(rhoL+rhoR)*(sigL+sigR)
            +2*sigL^2+2*sigR^2+2*sigL*sigR)/(2*a^2)];
            [V,D]=eig(R);
            lam1=D(1,1);
            lam2=D(2,2);
            L=inv(V)*[rhoL;sigL];
            R=inv(V)*[rhoR;sigR];
            uL=L(1);
            vL=L(2);
            uR=R(1);
            vR=R(2);
            if (lam1>=0)&&(lam2>=0)
                F=V*[lam1*uL;lam2*vL];
            elseif (lam1<0)&&(lam2>=0)
                F=V*[lam1*uR;lam2*vL];
            elseif (lam1<0)&&(lam2<0)
                F=V*[lam1*uR;lam2*vR];
            else
                F=V*[lam1*uL;lam2*vR];
            endif
            F1(j+1)=F(1);
            F2(j+1)=F(2);
        endif
    end
    for j =1:N
        rho(j,i) = rho(j,i-1) + dt/dx*(F1(j)-F1(j+1));
        sig(j,i) = sig(j,i-1) + dt/dx*(F2(j)-F2(j+1));
    end
  end
end
```