

The PRIMES 2015 Math Problem Set

Dear PRIMES applicant!

This is the PRIMES 2015 Math Problem Set. Please send us your solutions as part of your PRIMES application. For complete rules, see <http://web.mit.edu/primes/apply.shtml>

Note that this set contains two parts: “General Math problems” and “Advanced Math.” Please solve as many problems as you can in both parts.

You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF (preferred), DOC, or JPG file. The name of the attached file must start with your last name, for example, “smith-math-solutions.” Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.

You are allowed to use any resources to solve these problems, *except other people’s help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that posting these problems on problem-solving websites before the application deadline is not allowed. Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days. We encourage you to apply if you can solve at least 50% of the problems.

We note, however, that there will be many factors in the admission decision besides your solutions of these problems.

Enjoy!

General math problems

Problem G1. You roll three dice trying to get all three to be equal. If at some throw two are equal, you keep rolling the third one in the hope of it turning equal to the other two. What is the chance that you *do not* succeed if you are allowed to roll the dice two times? n times? (note that rolling all three dice or just rolling the third one each count as one roll).

Problem G2. John lives 2 miles north from a road, which is separated from John's house by a grove. If he walks from his house to the road along any straight line, the last mile of his walk is through the grove. Find the shape of the grove (describe its northern boundary by an equation).

Problem G3. Two white and two black rooks are placed at random on a standard 8-by-8 chessboard. What is the chance that NO white rook attacks a black rook? (recall that a rook attacks along the vertical and horizontal line it stands on). Give the answer as a fraction in lowest terms.

Problem G4. The number 1 is written on a blackboard. John plays the following game with himself: he chooses a number on the blackboard, multiplies it by 2 or 3, adds 1, and puts the result on the blackboard (if it does not appear there already).

- a) How many pairs of consecutive numbers can appear?
- b) Can it happen that three consecutive numbers appear on the blackboard?
- c) Can it happen that the numbers $n, n + 1, n + 3, n + 4$ appear for some n ?

Problem G5. Prove that for every real $C > 0$, there is some finite nonempty set of integers A such that $|A + A| \geq C|A - A|$ (where $|X|$ is the number of elements in a set X). Here

$$A + A := \{a + b : a \in A, b \in A\}$$

and

$$A - A := \{a - b : a \in A, b \in A\}$$

are the set of pairwise sums and the set of pairwise differences of A , respectively.

For example, let $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$. We have $A + A = [0, 28] \setminus \{1, 20, 27\}$ and $A - A = [-14, 14] \setminus \{\pm 6, \pm 13\}$. (Here $[m, n]$ denotes $\{m, m + 1, \dots, n\}$). Thus $|A + A| = 26$ and $|A - A| = 25$, so the example proves the statement for $C \leq 26/25$.

Problem G6. Let $a_n, n \geq 1$ be the sequence determined recursively by the rule

$$a_1 = 1, \quad a_{n+1} = \frac{n+2}{n+1}a_n + \frac{n^3 + 3n^2 + 2n - 2}{n(n+1)}.$$

Find a formula for a_n .

Hint. Compute the first few values and try to guess the answer.

Problem G7. Let M_n denote the $n \times n$ matrix whose entries are 0 below the main diagonal and 1 on and above the main diagonal. E.g.,

$$M_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Determine M_n^r for any $r \geq 1$.

Advanced math problems

Problem M1. There are $2k$ points on the plane. No three of them are co-linear. We build the following graph based on this configuration. Each point is a vertex. Two vertices are connected if and only if the line passing through them divides the other points into two equal sets: there are $(k - 1)$ points on each side of the line. We call this graph the middle-road graph based on the given configuration of points.

a) The complete bipartite graph $K_{n,m}$ has vertices $1, \dots, n + m$, and i, j are connected if and only if $i \leq n, j > n$, or $i > n, j \leq n$. For which pairs n, m , can the complete bipartite graph $K_{n,m}$ be the middle-road graph of some configuration?

b) The path graph P_n on n vertices is the graph with vertices $1, \dots, n$ such that i, j are connected if and only if $|i - j| = 1$. For which n , can the path graph P_n be the middle-road graph of some configuration?

If the graph exists, explain how to construct it. If not, prove that it doesn't exist.

Problem M2. Let $n \geq 2$ be an integer. You choose n points x_1, \dots, x_n uniformly at random on a circle of length 2. Let $0 < t \leq 1$. What is the chance that all the points belong to an arc of length t ?

Problem M3. (i) Let x_1, \dots, x_k be variables, and n_1, \dots, n_k be positive integers. Let A be the matrix with entries $a_{ij} := x_i^{n_j - 1}$, and $P(x_1, \dots, x_k) = \det A$. Show that

$$\frac{P(x_1, \dots, x_n)}{\prod_{i < j} (x_i - x_j)}$$

is a polynomial with integer coefficients.

(ii) Show that

$$\prod_{i < j} \frac{q^{n_i} - q^{n_j}}{q^i - q^j}$$

is a polynomial in q with integer coefficients. Deduce that

$$\prod_{i < j} \frac{n_i - n_j}{i - j}$$

is an integer.

Problem M4. Let f be a continuous function on the plane. In any rectangle $ABCD$ so that AB is parallel to the x -axis and B has greater y -coordinate than C , we have

$$\int_{ABC} f = \int_{CDA} f.$$

Prove that f is constant.

Problem M5. Two players, A and B, are playing a game with a special coin which, when tossed, is heads up with probability $2/3$ and tails up with probability $1/3$. The rules are as follows:

- First A chooses one pattern from the four patterns HH, HT, TH and TT.
- Next B chooses one pattern different from the one A chose.
- Then they toss the coin until one of their chosen patterns appear. That person will win the game.

For example, if A chose HH and B chose TT, and by tossing the coin they got HTHTHTT, then the winner would be B.

Now suppose A and B are both smart enough. Then what is the probability that A wins the game? What pattern should A choose? What pattern should B choose?

Problem M6.

Let V be a subset of a finite field L which is closed under addition (i.e, a subgroup of the additive group of L).

(a) Show that the polynomial $\prod_{v \in V} (X + v) \in L[X]$ is a p -polynomial

(that is, an L -linear combination of the monomials $X^{p^0} = X, X^{p^1} = X^p, X^{p^2}, \dots$).

(b) Let $t \in L \setminus V$. Prove that

$$\sum_{v \in V} \frac{1}{t + v} = \left(\prod_{v \in V} \frac{1}{t + v} \right) \cdot \left(\prod_{v \in V \setminus 0} v \right).$$