Classification of 7-Dimensional Unital Commutative Algebras

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Algebras

Let K be a field (e.g. $\mathbb{C}$).

Definition

A vector space A over K is called an **algebra** *over K if A is equipped with a product operation which is compatible with the addition and scalar multiplication.*

Finite Dimensional Algebras

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- An algebra A is said to be **finite dimensional** if A is finite dimensional as a vector space.
- An algebra A is said to be **unital** if there exists an identity $I \in A$ such that $I \cdot a = a \cdot I = a$ for all $a \in A$.
- An algebra A is called **commutative** if $ab = ba$ for all $a, b \in A$.

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- The algebra $M_n(\mathbb{C})$ is not commutative if $n > 1$.

Examples of Finite Dimensional Algebras

A: the algebra of all 3×3 matrices of the following form:

$$\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$$

for any $a, b, c \in \mathbb{C}$.

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A: a 3-dimensional unital commutative algebra over $\mathbb{C}$.

Homomorphisms of Algebras

A, B: two algebras over the field K.

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A map $f : A \to B$, *is called a* **homomorphism** *if*

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Isomorphisms of Algebras

- A homomorphism $f : A \to B$, is called an **isomorphism** if it is bijective.

- Two algebras A and B are said to be **isomorphic** if there exists an isomorphism $f : A \to B$.

Theorem

Let K be a field. If A is an n-dimensional unital algebra over K, then A is isomorphic to a subalgebra of $M_n(K)$, the unital algebra of all $n \times n$ matrices over K.

Problem

Classify unital finite dimensional commutative algebra up to isomorphism.

Discovering Examples Through Jordan Forms

Let A be a 7-dimensional unital commutative algebra over a field K. A is a subalgebra of $M_7(K)$. Assume that every element in A is upper triangular and A contains an element with the following Jordan form:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Jordan Form

For any x and y in A, we have $xJ = Jx$ and $yJ = Jy$. Hence

$$x = \begin{pmatrix} 0 & a & b & 0 & f & g & h \\ 0 & 0 & a & 0 & 0 & f & g \\ 0 & 0 & 0 & 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 & c & d & e \\ 0 & 0 & 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

Jordan Form

$$y = \begin{pmatrix} 0 & k & l & 0 & q & r & s \\ 0 & 0 & k & 0 & 0 & q & r \\ 0 & 0 & 0 & 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 & m & n & p \\ 0 & 0 & 0 & 0 & 0 & m & n \\ 0 & 0 & 0 & 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Jordan Form

$$xy = \begin{pmatrix} 0 & 0 & ak & 0 & 0 & aq+fm & ar+bq+fn+gm \\ 0 & 0 & 0 & 0 & 0 & 0 & aq+fm \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & mc & nc+md \\ 0 & 0 & 0 & 0 & 0 & 0 & mc \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Jordan Form

Using the above formula and $xy = yx$, we obtain

$$\frac{a-c}{f} = \frac{k-m}{q}.$$

Let

$$\alpha = \frac{a-c}{f} = \frac{k-m}{q}.$$

We have

$$\frac{\alpha r + n - l}{q} = \frac{\alpha g + d - b}{f}.$$

Let

$$\beta = \frac{\alpha r + n - l}{q} = \frac{\alpha g + d - b}{f}.$$

A New Family of Algebras

For each pair of (α, β), $A(\alpha, \beta)$ is the 7-dimensional unital commutative algebra of all matrices

$$x = \begin{pmatrix} k & a & b & 0 & f & g & h \\ 0 & k & a & 0 & 0 & f & g \\ 0 & 0 & k & 0 & 0 & 0 & f \\ 0 & 0 & 0 & k & c & d & e \\ 0 & 0 & 0 & 0 & k & c & d \\ 0 & 0 & 0 & 0 & 0 & k & c \\ 0 & 0 & 0 & 0 & 0 & 0 & k \end{pmatrix}$$

for all $k, a, b, c, d, e, f, g, h \in K$ satisfying

$$a - c = \alpha f, \qquad \alpha g + d - b = \beta f.$$

A Classification Problem

Problem

For two pairs of (α, β) *and* (α', β'), *when is* $A(\alpha, \beta)$ *isomorphic to* $A(\alpha', \beta')$?

Classification Theorem

Theorem

If $\alpha \neq 0$ and $\beta \neq 0$, then $A(\alpha, \beta)$ is isomorphic to the 7-dimensional unital commutative algebra:

$$\{k_0 I + k_1 x + k_2 x^2 + k_3 y + k_4 y^2 + k_5 y^3 + k_6 z : \quad k_i \in K\},$$

where x, y, z are the generators satisfying the relations:

$$x^3 = 0, \quad y^4 = 0, \quad z^2 = 0, \quad xy = xz = yz = 0.$$

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Corollary: If $\alpha \neq 0$, $\beta \neq 0$, $\alpha' \neq 0$ and $\beta' \neq 0$, then $A(\alpha, \beta)$ is isomorphic to $A(\alpha', \beta')$.

Proof of Theorem

Let

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & \frac{1}{\alpha} & \frac{\beta}{\alpha^2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{\alpha} & \frac{\beta}{\alpha^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\alpha} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$b = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{\alpha} & -\frac{\beta}{\alpha^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\alpha} & -\frac{\beta}{\alpha^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\alpha} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

Proof

$$c = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof

We have

$$a^3 = 0, \quad b^4 = 0, \quad c^2 = 0, \quad ab = ac = bc = 0.$$

We can verify that

$$A(\alpha, \beta) =$$

$$\{k_0 I + k_1 a + k_2 a^2 + k_3 b + k_4 b^2 + k_5 b^3 + k_6 c : \quad k_i \in K\}.$$

We construct an isomophism by: $a \to x, \quad b \to y, \quad c \to z$.

QED

Classification of Unital 7-dimensional Commutative Algebras

For any $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in a field K, define $A(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ to be the 7-dimensional unital commutative algebra over K:

$$\{k_0 I + k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4 + k_5 B_1 + k_6 B_2 : k_i \in K\},$$

where $x_1, x_2, x_3, x_4, B_1, B_2$ are the generators satisfying the relations:

(1) $x_i B_j = 0$ for all i and j;

(2) $B_i B_j = 0$ for all i and j;

(3) $x_i x_j = 0$ for all $i \neq j$;

(4) $x_i^2 = B_1 + \alpha_i B_2$ for all i;

(5) $x_i^3 = 0$ for all i.

A Classification Result

Theorem

Let K be an algebraically closed field. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ be scalars in K. Assume that $\alpha_i \neq \alpha_j$ for some pair i and j. The unital commutative algebras $A(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $A(\beta_1, \beta_2, \beta_3, \beta_4)$ are isomorphic if and only if there exists an invertible matrix $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ and a permutation σ of $\{1, 2, 3, 4\}$ such that

$$\beta_i = \frac{q_{21} + q_{22}\alpha_{\sigma(i)}}{q_{11} + q_{12}\alpha_{\sigma(i)}}.$$

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A field K is said to be *algebraically closed* if every polynomial equation with coefficients in K has a solution in K (e.g. $\mathbb{C}$).

Proof of the If Part (the Easy Part)

We denote the generators of $A(\beta_1, \beta_2, \beta_3, \beta_4)$ by:
$y_1, y_2, y_3, y_4, C_1, C_2$.
We construct an isomorphism:

$$f : A(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \to A(\beta_1, \beta_2, \beta_3, \beta_4)$$

as follows:

$$f(x_i) = \sqrt{q_{11} + \alpha_i q_{12}} \;\; y_{\sigma^{-1}(i)},$$

$$f(B_1) = q_{11} C_1 + q_{21} C_2,$$

$$f(B_2) = q_{12} C_1 + q_{22} C_2.$$

QED

A Consequence

The following result gives an easily verifiable necessary condition for two algebras in the family to be isomorphic.

Theorem

Assume that $\alpha_i \neq \alpha_j$ for some pair i and j. If $A(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is isomorphic to $A(\beta_1, \beta_2, \beta_3, \beta_4)$, then there exists a permutation σ of $\{1, 2, 3, 4\}$ such that

$$det \begin{pmatrix} \beta_1 & \beta_1\alpha_{\sigma(1)} & 1 & \alpha_{\sigma(1)} \\ \beta_2 & \beta_2\alpha_{\sigma(2)} & 1 & \alpha_{\sigma(2)} \\ \beta_3 & \beta_3\alpha_{\sigma(3)} & 1 & \alpha_{\sigma(3)} \\ \beta_4 & \beta_4\alpha_{\sigma(4)} & 1 & \alpha_{\sigma(4)} \end{pmatrix} = 0.$$

Proof

By the previous theorem, there exists an invertible matrix $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ and a permutation σ of $\{1, 2, 3, 4\}$ such that

$$\beta_i = \frac{q_{21} + q_{22}\alpha_{\sigma(i)}}{q_{11} + q_{12}\alpha_{\sigma(i)}}.$$

Proof

It follows that the following linear system (with q_{11}, q_{12}, q_{21}, and q_{22} as the unknowns) has a nonzero solution:

$$\beta_1\ q_{11} + \beta_1\alpha_{\sigma(1)}\ q_{12} - q_{21} - \alpha_{\sigma(1)}\ q_{22} = 0,$$

$$\beta_2\ q_{11} + \beta_2\alpha_{\sigma(2)}\ q_{12} - q_{21} - \alpha_{\sigma(2)}\ q_{22} = 0,$$

$$\beta_3\ q_{11} + \beta_3\alpha_{\sigma(3)}\ q_{12} - q_{21} - \alpha_{\sigma(3)}\ q_{22} = 0,$$

$$\beta_4\ q_{11} + \beta_4\alpha_{\sigma(4)}\ q_{12} - q_{21} - \alpha_{\sigma(4)}\ q_{22} = 0.$$

Proof

Hence

$$det \begin{pmatrix} \beta_1 & \beta_1\alpha_{\sigma(1)} & 1 & \alpha_{\sigma(1)} \\ \beta_2 & \beta_2\alpha_{\sigma(2)} & 1 & \alpha_{\sigma(2)} \\ \beta_3 & \beta_3\alpha_{\sigma(3)} & 1 & \alpha_{\sigma(3)} \\ \beta_4 & \beta_4\alpha_{\sigma(4)} & 1 & \alpha_{\sigma(4)} \end{pmatrix} = 0.$$

QED

We give a new proof of the following result due to Professor Poonen.

Corollary

Let K be an algebraically closed field. There exist infinitely many non-isomorphic 7-dimensional unital commutative algebras over K.

Sketch of Proof

Recall that any algebraically closed field has infinitely many elements. We choose $\alpha_1 = 1, \alpha_2 = \alpha, \alpha_3 = \alpha^2, \alpha_4 = \alpha^3$ and apply the previous theorem.

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- A more ambitious project: classify all finite dimensional unital commutative algebras up to isomorphism.

Thank You!

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