Tiling-Harmonic Functions

Yilun Du
Vyron Vellis (Graduate Student Mentor)
Sergiy Merenkov (Faculty Mentor)

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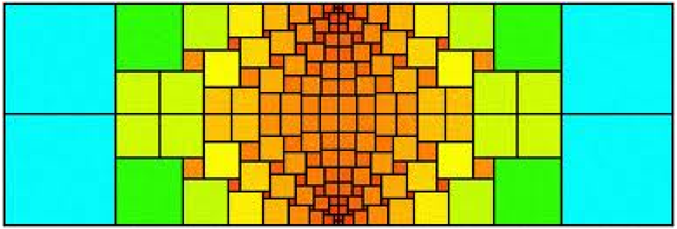
Introduction to Square Tilings

Definition (Square Tilings)

A square tiling T of a region D in $\mathbb{C}$ is a finite collection of squares with edges parallel to the x and y axes, that have mutually disjoint interiors and their union is all of D.



Definition of Energy

Suppose that u is a function defined on the vertices of the tiling T and t is a square in T.

Definition (The Oscillation of u on t)

$$\operatorname{osc}_u(t) = \max_p u(p) - \min_p u(p)$$

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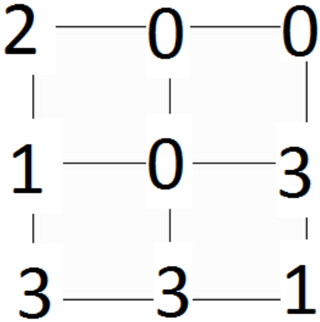
Definition (The Energy of u on T)

$$E_T(u) = \sum_{t \in T} \mathrm{osc}_u(t)^2.$$

A Standard Tiling Example



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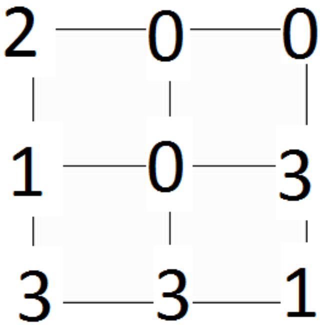

$$E = (2-0)^2 + (3-0)^2 + (3-0)^2 + (3-0)^2 = 31$$

Grid Harmonic Function

Definition (Grid(Tiling) Harmonic Function)

Suppose that u is defined on the boundary vertices of a tiling T. An extension of u is called harmonic if it minimizes the total energy. If we are given a infinite tiling T, such as a tiling of the upper real plane, then u is T-harmonic, if for any subtiling, u is harmonic on it.

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Theorem

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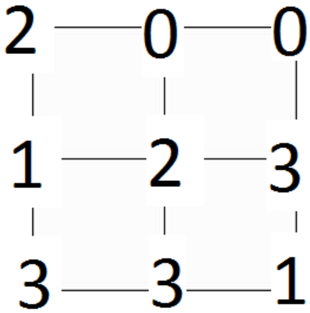
Remark: Energy minimizing functions are not unique.

An Standard Grid Tiling Energy Minimizing Example



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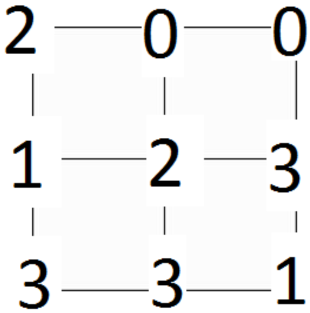

$$E = (2-0)^2 + (3-0)^2 + (3-1)^2 + (3-1)^2 = 21$$

Motivation for Tiling Harmonic Functions

- Tiling harmonic functions are analogs of carpet harmonic functions, harmonic functions defined on Sierpinski carpets.

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- Tiling harmonic functions are analogs of carpet harmonic functions, harmonic functions defined on Sierpinski carpets.
- If we can prove only non-negative T-harmonic functions u that vanish on the real vertices have the form $u(z) = cy$, where $c \geq 0$ is a constant, we might be able to generalize to carpet harmonic functions.
- Through this analog on carpet harmonic functions, we might be able to generate an alternative proof of the quasisymmetric rigidity of square Sierpinski carpets.

Grid Harmonic Algorithm

Step 1: We consider a 2×2 standard tiling, with tiles $S_1, \ldots, S_4$ where we want to minimize the inner point.

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Step 2: Let M_i and m_i be the maximum and minimum values of each of the S_i. We have the energy as

$$E_T(u) = (M_1 - m_1)^2 + (M_2 - m_2)^2 + (M_3 - m_3)^2 + (M_4 - m_4)^2$$

We sort all the M_i and m_i to be in increasing order. For example, we may have

$$m_1 < m_2 < M_1 < m_4 < M_2 < m_3 < M_3 < M_4,$$

We minimize the function in each interval.

Grid Harmonic Algorithm

Step 3: Suppose that we want to find the minimum energy of the function in an interval $[a, b]$

- If $b \leq m_i$ then let $E_i(x) = (M_i - x)^2$, $\alpha_i = 1$ and $c_i = M_i$.
- If $m_i \leq a \leq b \leq M_i$ then let $E_i(x) = (M_i - m_i)^2$, $\alpha_i = 0$ and $c_i = 0$.
- If $M_i \leq a$ then let $E_i(x) = (x - m_i)^2$, $\alpha_i = 1$ and $c_i = m_i$.

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- If $M_i \leq a$ then let $E_i(x) = (x - m_i)^2$, $\alpha_i = 1$ and $c_i = m_i$.

Step 4: If $\alpha_1 + \cdots + \alpha_4 = 0$ then let $X_i = a$. If $\alpha_1 + \cdots + \alpha_4 \neq 0$, then define

$$c = \frac{c_1 + c_2 + c_3 + c_4}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4},$$

and $E(x) = E_1(x) + E_2(x) + E_3(x) + E_4(x)$. If c is not between a and b then X_k is the one of a, b that minimizes E, or a if both have the same energy. If c is between a and b then X_k equals c.

Grid Harmonic Algorithm

Algorithm for General Tiling Grid

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Step 1: We process every interior point in our general grid and find the adjacent squares of the point. Assign values to all the interior points. Randomly generate some order of interior points to loop through.

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Algorithm for General Tiling Grid

Step 1: We process every interior point in our general grid and find the adjacent squares of the point. Assign values to all the interior points. Randomly generate some order of interior points to loop through.
Step 2: Process all interior points in our grid find the minimum value of each interior point with respect to surroundings.
Step 3: Run the algorithm again until the energy does not change significantly.

Grid Harmonic Algorithm

Problems

8	24	9	18	21	23	19	28	15	29
27	9	9	14	14	14	14	14	14	22
20	9	8.625	8.625	8.625	8.625	8.625	8.625	7.5	2
15	9	8.5	8.25	8.25	8.25	8.25	8.0625	7.5	1
11	9	8.25	8	7.875	7.875	7.875	7.875	7.5	21
0	7.5	7.5	7.5	7.5	7.5	7.5	7.5	7.5	27
26	13	10	25	9.33333	9.33333	9.333333	9.33333	9.33333	4
27	13	11.1667	11.1667	11.1667	11.1667	11.1667	11.1667	13	29
25	13	13	13	13	13	13	13	13	29
13	20	29	15	20	26	28	18	20	4

8	24	9	18	21	23	19	28	15	29
27	19	14	14	14	14	14	14	14	22
20	19	14	14.25	14.25	14.25	14.25	14.25	14.25	2
15	15	14	14.5	14.5	14.5	14.5	14.5	14.25	1
11	11	14	14.825	15.2	15.3333	15.75	17	20.625	21
0	11	14	15	15.65	15.9	16.1667	17	20.625	27
26	19	17	15	16.15	16.475	16.6	17	17	4
27	20	17.6	15	17.3	17.3	17.3	17.3	18	29
25	20	20	15	20	20	20	18	18	18
13	20	29	15	20	26	28	18	20	4

Grid Harmonic Algorithm

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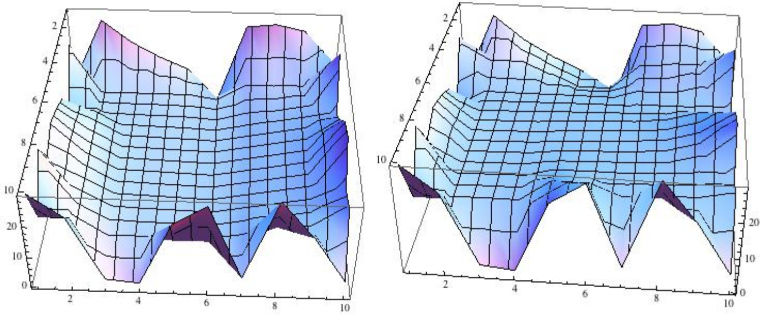

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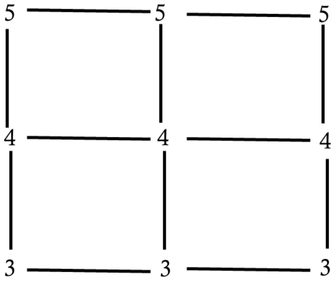
Theorem

There are only a finite number of local minima of energy.

Graph Harmonic Function

Definition (Graph Harmonic Function)

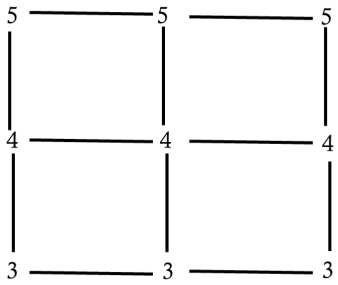
A function $u \in \mathcal{F}(T)$ is called **graph-harmonic** if for every interior vertex p of T, the value $u(p)$ is equal to the average of u on the neighbor vertices of p.



Graph Harmonic Function

Theorem

A function $u \in \mathcal{F}(T)$ is also called **graph-harmonic** *if the sum of the squares of neighboring vertices is minimized.*



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Step 2: We then loop through all interior points and set each one equal to the average of its neighbors.

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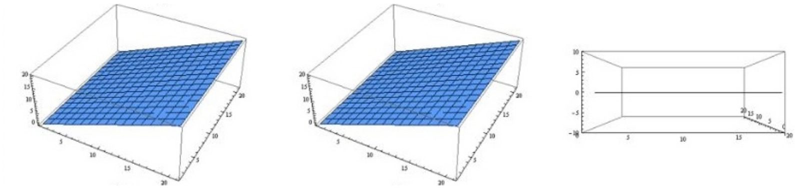
Step 2: We then loop through all interior points and set each one equal to the average of its neighbors.

Step 3: We repeat until values do not change significantly.

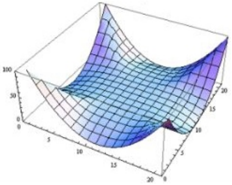
Visual Representation

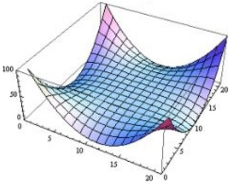
Example 1: $u(0, j) = j,\ u(20, j) = j,\ u(0, i) = 0,\ u(20, i) = 20.$

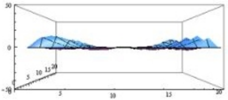


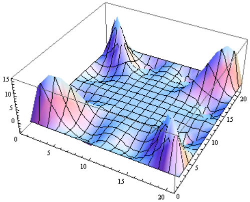
Visual Representation

Example 2: $u(0,j) = (10-j)^2$, $u(20,j) = (10-j)^2$, $u(0,i) = (10-i)^2$, $u(20,i) = (10-i)^2$.



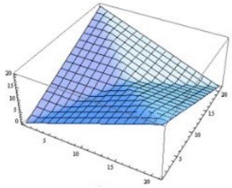


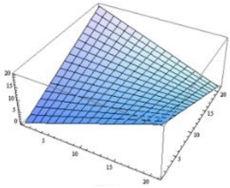


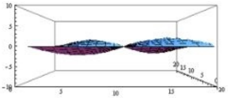


Visual Representation

Example 3: $u(0,j) = j$, $u(20,j) = 20 - j$, $u(0,i) = i$, $u(20,i) = 20 - i$.







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- Given that all 2×2 grids have their interiors minimized, it does not follow that the grid containing the tiles has minimum energy.
- Given plane boundary conditions both tiling and grid harmonic lie on the plane.

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- Is it true that all planes are grid harmonic?
- Is it true that the only bounded T-harmonic functions are constant?
- If T is a tiling of the upper complex plane and u vanishes on the real vertices and is T-harmonic, is it true that $u(z) = cy$?

Acknowledgments

- PRIMES Program
- Illinois Geometry Lab
- Mentor: Dr. Merenkov
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