Cellular Automata on a Hexagonal Grid

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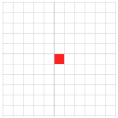
Table of Contents

Cellular Automaton Rule

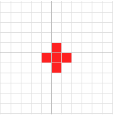
Rule: A cell is born if it is adjacent to exactly one live cell. A live cell never dies.
Initial conditions: A single live cell at the origin.

Growth on Square Grid

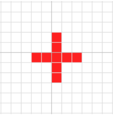
Figure: First Six generations



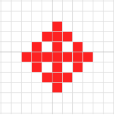
(a) Generation 0



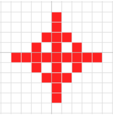
(b) Generation 1



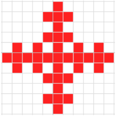
(c) Generation 2



(d) Generation 3



(e) Generation 4



(f) Generation 5

Growth on Square Grid (continued)

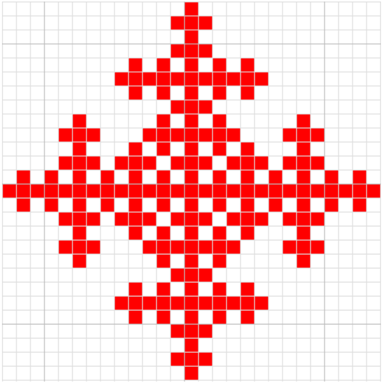


Figure: Growth after 13 generations

Square Grid Questions

Two major questions:

Square Grid Questions

Two major questions:

- Which cells are born?

Square Grid Questions

Two major questions:

- Which cells are born?
- In what generation are they born?

Square Grid Answers

Theorem

A point (x, y) is born if and only if the highest power of 2 dividing x is not equal to the highest power of 2 dividing y.

Square Grid Answers

Theorem

A point (x, y) is born if and only if the highest power of 2 dividing x is not equal to the highest power of 2 dividing y.

Theorem

For a point (x, y), let 2^k be the largest power of 2 less than $|x| + |y|$. Then we can recursively define $f(x, y)$, the generation in which (x, y) is born, as

$$f(x,y) = \begin{cases} 2^k + f(\max(|x|,|y|) - 2^k, \min(|x|,|y|)) & |x| \neq |y| \\ \infty & |x| = |y| \neq 0 \\ 0 & |x| = |y| = 0. \end{cases}$$

Table of Contents

Hexagonal Grid Rules

Rule: A cell is born if it is adjacent to exactly one live cell. A live cell never dies.
Initial conditions: A single live cell at the origin.

Golly Software

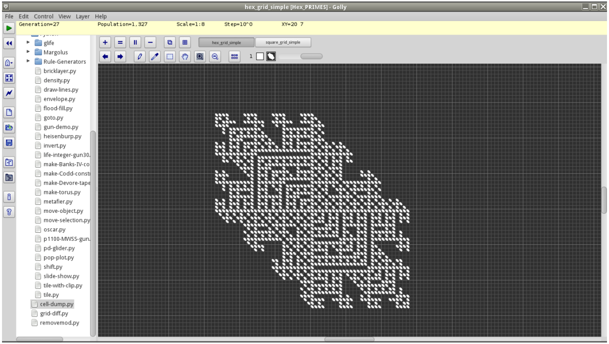
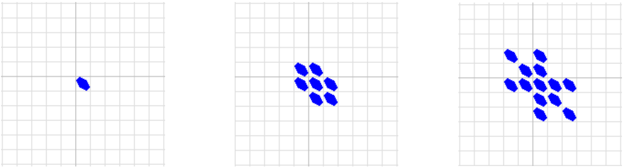
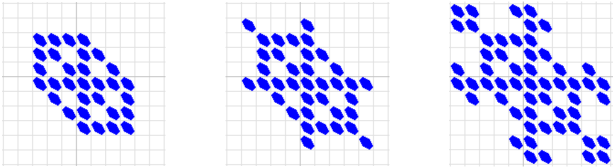


Figure: Golly simulation software

Growth on Hexagonal Grid



(a) Generation 0 (b) Generation 1 (c) Generation 2



(d) Generation 3 (e) Generation 4 (f) Generation 5

Growth on Hexagonal Grid

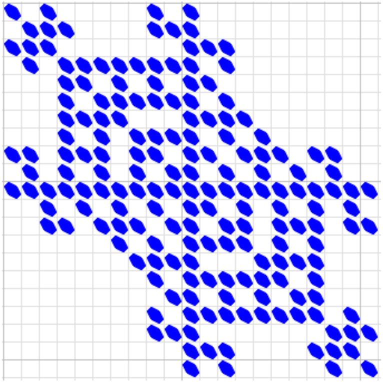


Figure: Growth after 10 generations

Growth on Hexagonal Grid

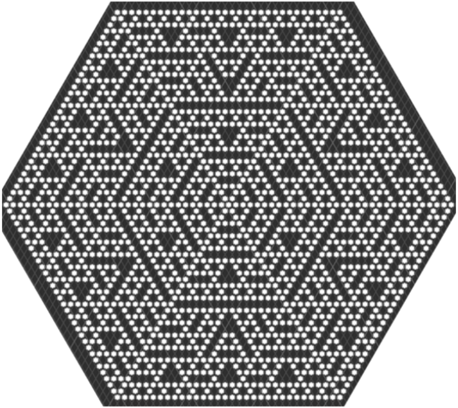


Figure: Growth after 31 generations

Symmetries

Symmetries

- $y = x$
- Rotational about origin

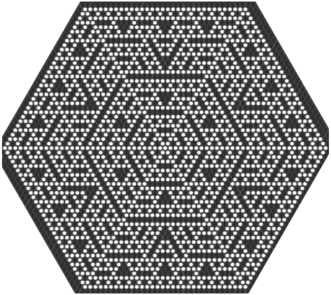


Figure: Growth after 31 generations

Lineage

Definition

Parent: the live cell which caused another cell to be born by being adjacent to it.

Definition

Lineage: the sequence of live cells from the origin to any live cell such that each cell is the parent of the next one.

Pioneers

Definition

Pioneer: a point (x, y) which is born in generation $x + y$.

Sierpinski Sieve

Lemma

The set of all pioneers is equal to the Sierpinski sieve

- This gives pioneers a simple, recursive structure

Sierpinski Sieve

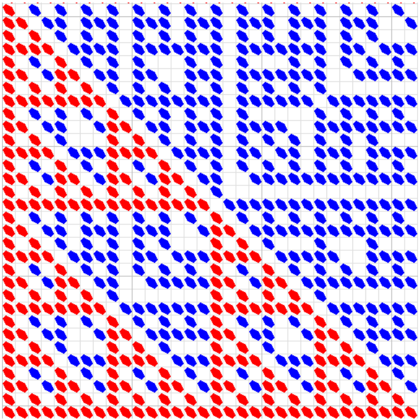
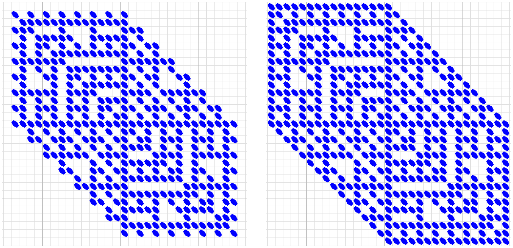


Figure: Overlay of Sierpinski Sieve on Hexagonal Grid

Complete Generations

All the points (x, y) with $x + y = 2^n - 1$ are born



(a) Generation 14 (b) Generation 15

Recursive Formula

Theorem

Given a point (x, y), there exists some $k \in \mathbb{N}$ such that $2^k \leq x + y < 2^{k+1}$. Assume without loss of generality that $x \geq y$. The generation in which a cell (x, y) is born is given by:

$$f(x,y) = \begin{cases} f(x - 2^k, y) + 2^k & x \geq 2^k \\ f(x + y - 2^k - 1, 2^k - x) + 2^k + 1 & x < 2^k. \end{cases}$$

Recursive Structure

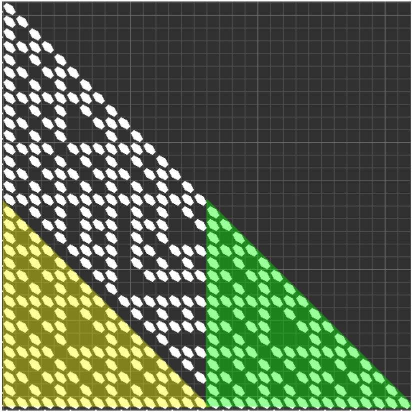


Figure: Case 1: $f(x, y) = f(x - 2^k, y) + 2^k$

Recursive Structure cont'd

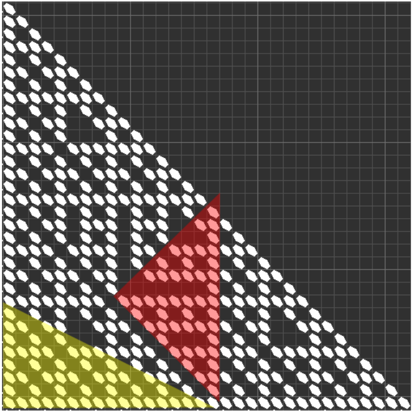


Figure: Case 2: $f(x, y) = f(x + y - 2^k - 1, 2^k - x) + 2^k + 1$

Overlay of Square and Hexagonal Grid

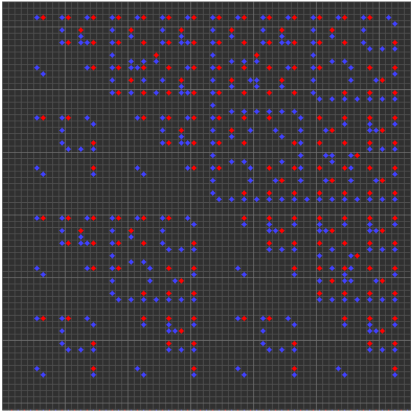


Figure: XOR of Square and Hexagonal Grid

Difference of Grids

- Difference is much sparser than individual grids
- Red points have a much simpler structure than blue points

Future Research

Goals:

- Closed formula to determine whether a point is born
- Complete proof of recursive formula
- Determine population at any time

Acknowledgements

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- Our mentor Tanya Khovanova
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- Our parents