The PRIMES 2013 problem set

Dear PRIMES applicant,

This is the PRIMES 2013 problem set. Please send us your solutions as part of your PRIMES application by December 1, 2012. For complete rules, see http://web.mit.edu/primes/apply.shtml

Note that this set contains three sections: "General math problems" (for all tracks) and two sections corresponding to the two research tracks of PRIMES 2013 ("Advanced math" and "Computer science"). Please solve as many problems as you can in the General math section, and also in the section(s) corresponding to the track(s) for which you are applying.

You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF (preferred), DOC, or JPG file. The name of the attached file must start with your last name, for example, "smith-solutions." Include your full name in the heading of the file.

Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.

You are allowed to use any resources to solve these problems, *except other people's help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day, and think about them, on and off, over some time (several days). We encourage you to apply if you can solve at least 50% of the problems. [1]

Enjoy!

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[1] We note, however, that there will be many factors in the admission decision besides your solutions of these problems.

General math problems

Problem G1. You toss two coins trying to get two heads. Namely, you toss both coins until you get at least one head. If you get two heads at that point, you finish, and if you get just one, you keep tossing the other coin until you get a head.

(a) What's the chance that you will not finish in m tosses? (Tossing two coins simultaneously counts as one toss).

(b) Give the answer as a fraction for $m = 8$.

Solution. One way it can happen is if we don't get any heads in the first m tosses. The probability of this is $1/4^m$. The other way is if we get a single head in the k-th toss, $1 \le k \le m$, (the probability of which is $1/2 \cdot 4^{k-1}$) but the other coin does not give a head on tosses $k+1, ..., m$ (probability 2^{k-m}). Adding up, we get $1/2^{m-1} - 1/4^m = \frac{2^{m+1}-1}{4^m}$. For $m = 8$ we get $511/65536$.

Problem G2. A black Bishop and a white King are placed on a $2n$ by $2n$ chessboard (all positions are equally likely). What is the chance that the Bishop attacks the King (i.e., the King is under check)? (Recall that the Bishop strikes along diagonals). Write the answer as a rational function of n.

Solution. The bishop hits $2n - 1 + 2m$ cells if it is m positions from the boundary. So we get

$$p = \frac{2n-1}{4n^2-1} + \sum_{m=1}^{n} \frac{2(n-m)^2}{n^2(4n^2-1)} =$$

$$\frac{2n-1}{4n^2-1} + \frac{n(n-1)(2n-1)}{3n^2(4n^2-1)} = \frac{8n^2-6n+1}{3n(4n^2-1)} = \frac{4n-1}{3n(2n+1)}.$$

Problem G3. A sequence $a_n, n \ge 0$ of positive integers satisfies the recursion $a_{n+1} = F(a_n)$, where $F(x)$ is the sum of the squares of all the digits of x (for example, $F(324) = 3^2 + 2^2 + 4^2 = 29$). The number a_0 can be arbitrary.

(a) Show that either $a_n = 1$ starting from some n, or a_n is periodic with minimal period 8 starting from some n. What are the 8 numbers repeating periodically in the second case?

(b) Let h be any nonnegative integer function on the set of digits $0, 1, ..., 9$. Consider the recursion $b_{n+1} = F_h(b_n)$, where $F_h(x) := \sum_i h(x_i)$, and x_i are the digits of x. Show that b_n is eventually periodic, and there are finitely many possible minimal periods for it.

Solution. (a) It is easy to check that for small initial values, the sequence becomes eventually 1 or hits the 8-cycle $89, 145, 42, 20, 4, 16, 37, 58$. But for large x (bigger than 99), $F(x) < x$. So the result follows.

(b) Let $N = \max(F_h)$. Then $F(x) \leq N(1 + \log_{10} x)$. So if $x > N(1 + \log_{10} x)$ (which is true for large enough x) then $F(x) < x$. Let S be the set of x such that $x \leq N(1 + \log_{10} x)$, and let M be the maximum of $S \cup F(S)$. Then the set T of numbers $x \leq M$ is stable under F, and b_n is eventually in T (as at some point it will hit S). On T, there are finitely many cycles, and eventually the sequence b_n will hit one of them.

Problem G4. Let $a > 1$.

(a) For which a does there exist a solution $0 \leq x \leq 1$ of the equation
$$(1 + x)^{1/3} + (1 - x)^{1/3} = a?$$

Show that if such a solution exists, it is unique, and find it.

(b) For which a does there exist a solution $0 \leq x \leq 1$ of the equation
$$(1 + x)^{1/4} + (1 - x)^{1/4} = a?$$

Show that if such a solution exists, it is unique, and find it.

Solution. (a) We get, taking the cube:
$$2 + 3a(1 - x^2)^{1/3} = a^3.$$

So $(1 - x^2)^{1/3} = (a^3 - 2)/3a$, which must be nonnegative and ≤ 1, so $2^{1/3} \leq a \leq 2$. Then $1 - x^2 = \left(\frac{a^3 - 2}{3a}\right)^3$, so
$$x = \sqrt{1 - \left(\frac{a^3 - 2}{3a}\right)^3}.$$

(b) We get by raising to 4th power:
$$2 + 4((1 + x)^{1/4} + (1 - x)^{1/4})^2(1 - x^2)^{1/4} - 2(1 - x^2)^{1/2} = a^4.$$

So for $z = (1 - x^2)^{1/4}$ we have
$$2z^2 - 4a^2 z + a^4 - 2 = 0.$$

Thus $z = \frac{4a^2 - \sqrt{16a^4 - 8(a^4 - 2)}}{4} = a^2 - \sqrt{a^4/2 + 1}$ (the sign is $-$ since $z \leq 1$). So $1 - x^2 = (a^2 - \sqrt{a^4/2 + 1})^4$, and
$$x = \sqrt{1 - (a^2 - \sqrt{a^4/2 + 1})^4}.$$

The range for a is between $2^{1/4}$ and 2.

Problem G5. Find all rational numbers a, b for which the number $\gamma = \log_2(a + b\sqrt{2})$ is rational.

Solution. Clearly, this number is rational if one of the numbers a, b is an integer power of 2 and the other is zero. Let us show that

otherwise γ is irrational. Assume $\gamma = m/n$. Then
$$2^m = (a + b\sqrt{2})^n.$$
Passing to conjugates, $2^m = (a - b\sqrt{2})^n$, i.e. $a + b\sqrt{2} = \pm(a - b\sqrt{2})$. So $a = 0$ or $b = 0$, so the other number must be positive. In the first case we have $a^n = 2^m$, so a is a power of 2. The second case is similar (b is a power of 2).

Advanced math problems

Problem M1. You toss a coin n times. What is the probability that the number of heads you'll get is divisible by 3? (Find an exact formula, not involving sums of unbounded length; it may depend on the remainder of n modulo 6).

Solution. We have $p = 2^{-n}(1 + \binom{n}{3} + \binom{n}{6} + ...)$. This can be written as

$$p = \frac{(1+z)^n + (1+z^{-1})^n + 2^n}{3 \cdot 2^n},$$

where z is a cubic root of 1. This implies that

$$p = \frac{w^n + w^{-n} + 2^n}{3 \cdot 2^n},$$

where w is a 6th root of 1. So we get that

$$p = \frac{1}{3} + \frac{b_n}{3 \cdot 2^n},$$

where $b_n = 2, 1, -1, -2, -1, 1$ if $n = 0, 1, 2, 3, 4, 5$ modulo 6, respectively.

Problem M2. (a) Let $c < 2\pi$ be a positive real number. Show that there are infinitely many integers n such that the equation

$$x^2 + y^2 + z^2 = n$$

has at least $c\sqrt{n}$ integer solutions.

(b) Find a constant $C > 0$ such that there are infinitely many n for which the equation

$$x^5 + y^3 + z^2 = n$$

has $\geq Cn^{1/30}$ nonnegative solutions.

Solution. (a) Let B_n be the number of solutions of $x^2 + y^2 + z^2 \leq n$. It is easy to see that $B_n \sim \frac{4}{3}\pi n^{3/2}$ as $n \to \infty$ (volume of the ball). This means that $B_n - \frac{2}{3}cn^{3/2} \to +\infty$. So there are infinitely many n for which $B_n - B_{n-1} \geq \frac{2}{3}c(n^{3/2} - (n-1)^{3/2})$. But $\frac{2}{3}c(n^{3/2} - (n-1)^{3/2}) \sim cn^{1/2}$ as $n \to \infty$. This implies the statement.

(b) The solution is similar. Indeed, the number of nonnegative integer solutions of $x^5 + y^3 + z^2 \leq n$ behaves as the volume of the region defined by this inequality in the first octant. This volume is easy to compute: after making dilations $x \mapsto n^{1/5}x$, $y \mapsto n^{1/3}y$, $z \mapsto n^{1/2}z$, this region maps to $x^5 + y^3 + z^2 \leq 1$. Let V be the volume of the latter. Then the volume of the former is $Vn^{\frac{1}{5}+\frac{1}{3}+\frac{1}{2}} = Vn^{\frac{31}{30}}$. So we can take $C < (31/30)V$.

Problem M3. A finite string of 0s and 1s is called **admissible** if it occurs in one of the rows of the Pascal triangle modulo 2. I.e., a string

is admissible if it has the form

$$\binom{n}{m}, \binom{n}{m+1}, ..., \binom{n}{m+k},$$

where the binomial coefficients are taken modulo 2.

(a) Which strings of length ≤ 4 are not admissible? Why?

(b) Give an explicit description of all admissible strings of length n.

(c) What is the number $a(n)$ of admissible strings of length n? (Write a recursion, guess the answer and prove it by induction; the recursive formula may be different for $n = 2r$ and $n = 2r + 1$).

Solution. (a) It is easy to see that all strings but 1011 and 1101 are admissible. To show that these strings are not admissible, note that such a string cannot occur in an even row (since those have 0 at every other place). But $\binom{2r+1}{k+1} = \binom{2r+1}{k}\frac{2r+1-k}{k+1}$, so if k is odd, these numbers have the same parity. So every other element in a string occuring in an odd row is followed by the same element. But this is not satisfied for the two strings in question.

(b),(c) Assume for simplicity that n is even: $n = 2r$. Then there are the following kinds of admissible strings of length n.

1) (even rows) $0b_1....0b_r$, where $b_1...b_r$ is an admissible string of length r. The number of such is $a(r)$.

2) (even rows) similarly, $b_1 0...b_r 0$. The number of such is also $a(r)$.

3) (odd rows) $b_1 b_1...b_r b_r$. The string above of length $2r + 1$ would then be $0c_1 0c_2...0c_r 0$, where $c_1...c_r$ is admissible of length r. There are $a(r)$ of such strings.

4) (odd rows) $b_0 b_1 b_1....b_{r-1} b_{r-1} b_r$. The string above is $c_1 0c_2 0...0c_{r+1}$, where $c_1...c_{r+1}$ is admissible. The number of such strings is $a(r+1)$.

So at first sight we get the equation

$a(2r) = 3a(r) + a(r+1)$.

But it is not quite right because the classes (1)-(4) have intersections. Namely, the string of zeros belongs to all 4, and the string of ones to the last two, and 0...01 and 10...0 belong to 1) and 4) and 2) and 4) respectively. So we overcount by 6, and the correct equation is

$a(2r) = 3a(r) + a(r+1) - 6$.

In a similar way we get $a(2r + 1) = 2a(r) + 2a(r+1) - 6$.

Then using that $a(0) = 1$, $a(1) = 2$, $a(2) = 4$, it is easy to show by induction that $a(n) = n^2 - n + 2$.

Problem M4. Positive solutions of the equation $x \sin(x) = 1$ form an increasing sequence $x_n, n \geq 1$.

(a) Find the limit

$$c_1 = \lim_{n \to \infty} n(x_{2n+1} - 2\pi n).$$

(b) Find the limit
$$c_2 = \lim_{n \to \infty} n^3 \left(x_{2n+1} - 2\pi n - \frac{c_1}{n} \right).$$

Solution. (a) Let $x_{2n+1} = 2\pi n + b_n$. Clearly, $b_n \to 0$, and $\sin b_n = \frac{1}{2\pi n + b_n}$. This shows that $b_n \sim 1/2\pi n$, so the required limit is $c_1 = 1/2\pi$.

(b) We have $\sin x = x - x^3/6 + \ldots$. Thus, we get that up to quartic terms in $1/n$,

$$b_n = \frac{1}{b_n + 2\pi n} + \frac{b_n^3}{6} = \frac{1}{(2\pi n)^{-1} + 2\pi n} + \frac{1}{6(2\pi n)^3} = \frac{1}{2\pi n} - \frac{5}{6(2\pi n)^3}.$$

So $c_2 = -5/48\pi^3$.

Problem M5. Let us say that a polynomial f with complex coefficients is **degenerate** if there exists a square matrix B such that $B \neq f(A)$ for any square matrix A. What are all the degenerate polynomials of degree 2? degree 3? degree 4? any degree?

Solution. We claim that f is degenerate if and only if there exists $b \in \mathbb{C}$ such that $f(z) = b$ implies $f'(z) = 0$. Such polynomials have the form $f(z) = b + (z - a_1)^{n_1} \ldots (z - a_k)^{n_k}$, where $n_i > 1$ for all i. So for degree 2, all polynomials are degenerate; for degree 3, the degenerate polynomials are those with a degenerate critical point, and for degree 4, the degenerate polynomials are those of the form $b + g(z)^2$, where g is a quadratic polynomial.

Indeed, if f is of this form then the matrix $\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$ is not of the form $f(A)$. Indeed, A could not be diagonalizable, so it is conjugate to a Jordan block with some eigenvalue z. But then $f(A) = b \cdot \mathrm{Id}$, as $f'(z) = 0$. On the other hand, suppose that f is not of the above form. Let $J_{b,n}$ be the Jordan block of size n with eigenvalue b. Pick z such that $f(z) = b$ and $f'(z) \neq 0$. Then f is invertible near z. Let $A = f^{-1}(J_{b,n})$. This makes sense, as f^{-1} has a Taylor expansion $f^{-1}(w) = z + c_1(w - b) + c_2(w - b)^2 \ldots$. And we have $f(A) = J_{b,n}$. Then by the Jordan normal form theorem, for any B there is A such that $f(A) = B$.