Depths of Posets Ordered by Refinement

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  2. If $A \leq B$ and $B \leq A$, then $A = B$ (anti-symmetry);
  3. If $A \leq B$ and $B \leq C$, then $A \leq C$ (transitivity).

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- Posets may be represented by Hasse diagrams , in which elements A and B are connected, with A below B, if $A < B$ and there is no element C such that $A < C < B$.

Examples

- The set of the first 6 natural numbers, ordered by divisibility.

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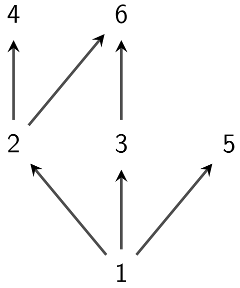
Hasse Diagrams



Examples

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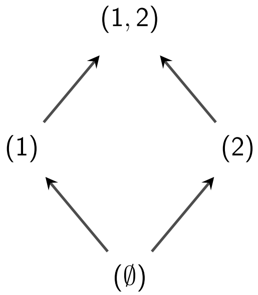

Intervals

- An interval $I = [A, B]$ of a poset includes all elements C such that $A \leq C \leq B$.

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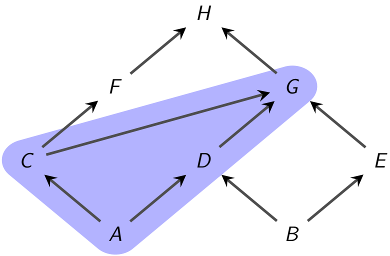
Example

The color blue represents the interval.

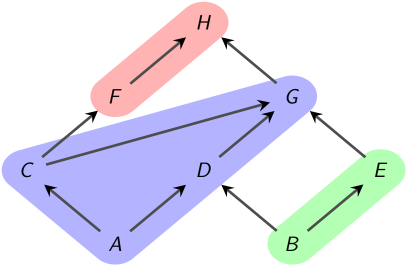


An interval $I = [A, G]$

Interval partitions of a poset

- In an interval partition, the poset is completely partitioned into non-overlapping intervals. Each element of the poset is in exactly one interval.

Example



An interval partition P

Depth

Definition

The *depth* of an element X_0 in a poset is defined to be the maximum possible number of elements in a chain $X_0 > X_1 > X_2 > \cdots > X_n$ from X_0 to the bottom of the poset.

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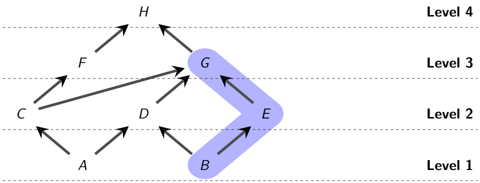
- We say that a *level n* contains all elements of depth n.

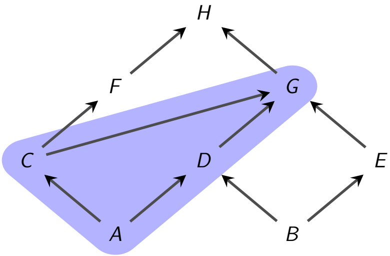
Example



nDepth

For an interval I, $ndepth[I] = max(depth[X])$ over all elements X in I.

- For $I = [A, B]$, $ndepth[I] = depth[B]$.

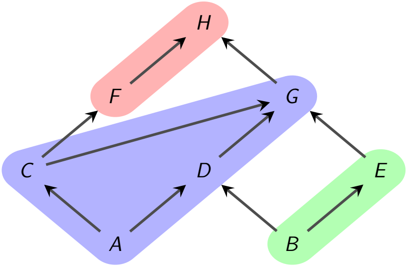


Interval $I = [A, G]$. $ndepth[I] = depth[G] = 3$.

nDepth

For a partition P of a poset, $ndepth[P] = min(ndepth[I])$ over all intervals I in P.

- Which interval in the partition has the least depth?

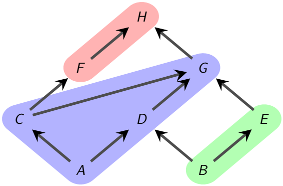


$$ndepth[P] = ndepth[B, E] = 2.$$

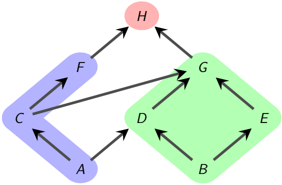
nDepth

For a poset G, $ndepth[G] = max(ndepth[P])$ over all possible partitions P of G.

- Which partition(s) of the poset have the greatest depth?



ndepth = 2



ndepth = 3

$ndepth[G] = 3.$

Past work

Yinghui Wang:

- ndepth of a product of chains $n^k \setminus 0$ is $(n-1)\lceil k/2 \rceil$

Biro, Howard, Keller, Trotter, and Young

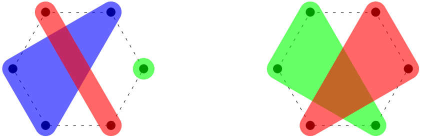
- For a poset B of the non-empty subsets of an n-element set ordered by inclusion, $ndepth[B] \geq n/2$

We will study the properties of posets comprised of partitions of sets ordered by refinement.

Set-Partitions

- A partition of a set is the division of the set of distinct points into subsets.
- Every element of the set is partitioned into some subset, and no element is within two or more subsets.

Examples

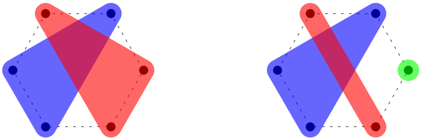


Partitions of a 6-element set.

Refinement

- It is possible to order partitions of a set by *refinement*.
- A partition of a set P_b is considered *finer* than another partition P_a if all subsets within P_b are within some subset in P_a. If P_b is finer than P_a, P_a is *coarser* than P_b.

Examples



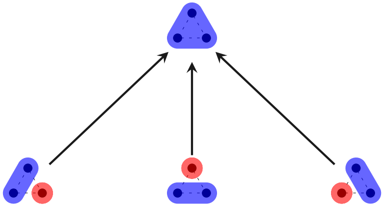
Right partition is finer than left partition

Posets of the refinement ordering

- These posets depend solely on the size of the set that is partitioned

Let G_i denote a poset ordered by refinement of all set partitions of the set with i elements except the "empty partition", the partition of each element into a separate subset.

Example



G_3.

Now that we know what a poset ordered by refinement looks like . . .

How do we find the ndepth of such a poset?

First, we need a few tools.

Rotations

- A rotation of a certain partition is any other partition that may be obtained by rotating the partition around a circle.
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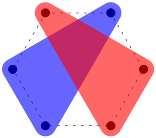

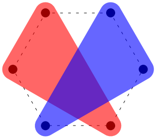

Classes of set partitions

- It is possible to group set partitions into classes based on the sizes of subsets in the set partitions.
- The class $C = (S_1, S_2, S_3, \ldots, S_n)$ with all S_i positive integers and $S_1 \geq S_2 \geq S_3 \geq \ldots \geq S_n$ includes all partitions which consist of exactly n subsets of size $S_1, S_2, \ldots, S_n$.
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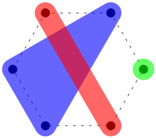
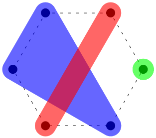
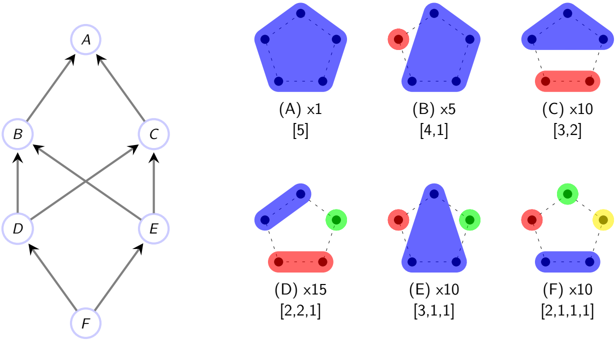




$ndepth[G_5]$



Poset of classes in G_5

$ndepth[G_5]$

In order for it to be true that $ndepth[G_5] = L \ldots$

- It must be impossible to partition the poset so that all intervals I have $ndepth[I] > L$
- It must be possible make a partiton P of G_5 such that each interval I in P has $ndepth[I] \geq L$

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Lemma

$ndepth[G_5] < 4$.

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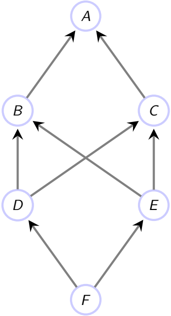
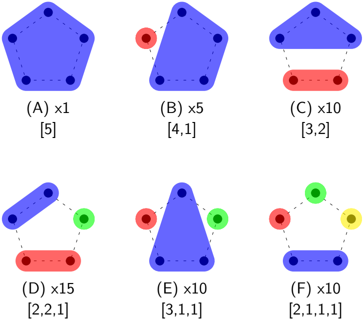




Poset of classes in G_5

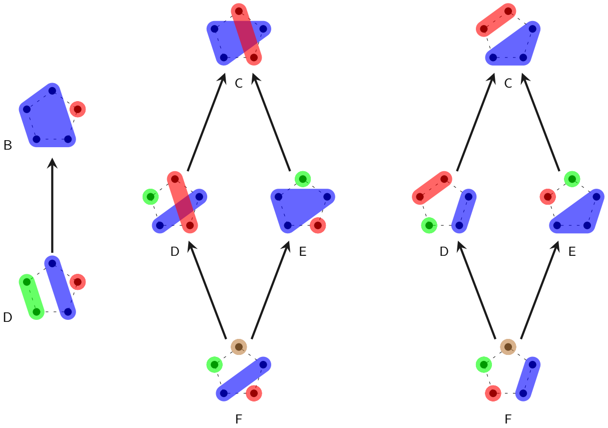
$ndepth[G_5]$

- We attempt to show that $ndepth[G_5] = 3$ by constructing a partition P of G_5 with ndepth 3.
- It is possible to make a partition P with ten non-overlapping intervals of the form $[F, C]$ and five non-overlapping intervals of the form $[D, B]$, as well as an interval $[A, A]$, which will completely partition the poset into intervals of ndepth 3 or greater.

Theorem

The ndepth of G_5 is 3.

$ndepth[G_5]$



The 10 intervals of the form $[F, C]$ and 5 intervals of the form $[D, B]$ include all the rotations of these three intervals.

Results

The values of $ndepth[G_i]$

i	1	2	3	4	5	6
$ndepth[G_i]$	-	1	1	2	3	3

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- $ndepth[G_i]$ increases infinitely; for any ndepth L, there is some i large enough that $ndepth[G_i] = L$.

Future Work

- Describe upper and lower bounds for $ndepth[G_i]$ for very large i
- Prove that $ndepth[G_i]$ increases linearly
- Concretely describe the sequence $ndepth[G_i]$

Acknowledgments

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- My mentor, Sergei Bernstein, for his patience and guidance
- Professor Richard Stanley, for suggesting the project
- Tanya Khovanova and Dai Yang, for their advice
- PRIMES, for providing this research opportunity
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