Good Functions and Multivariable Polynomials

Saarik Kalia and Michael Zanger-Tishler

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Good Functions

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- Later, the Sprindžuk-Baker Conjecture was proved by Margulis and Kleinbock, our project advisor, using a quantitative version of Margulis's method, and a key ingredient was the use of (C, α)-good functions.

Definition

- For $C, \alpha > 0$, a function $f : \mathbb{R}^n \to \mathbb{R}$ is (C, α)-good if for every ball $B \subset \mathbb{R}^n$ and $\epsilon > 0$,

$$\lambda_n(B^{f,\epsilon}) \leq C \left(\frac{\epsilon}{\|f\|_B} \right)^\alpha \lambda_n(B).$$

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- $\|f\|_B := \sup_{x \in B} |f(x)|$.
- $B^{f,\epsilon} := \{x \in B : |f(x)| < \epsilon\}$.
- If $\|f\|_B = 0$, we let $\frac{1}{0} = \infty$.
- λ_n denotes the Lebesgue measure in $\mathbb{R}^n$.

Example

$$\boxed{\lambda_n(B^{f,\epsilon}) \leq C \left(\frac{\epsilon}{\|f\|_B} \right)^{\alpha} \lambda_n(B)}$$

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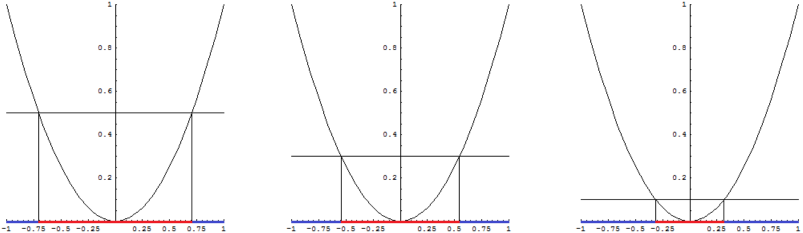

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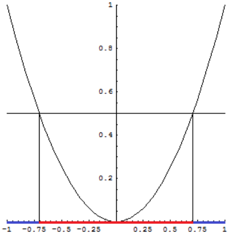

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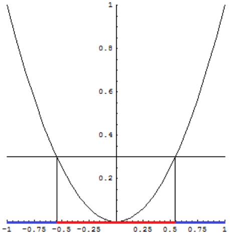
$\lambda_1(B^{f,\epsilon}) = \sqrt{2}$



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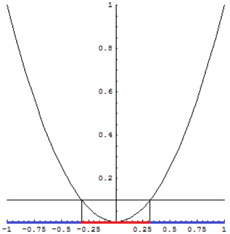
$\lambda_1(B^{f,\epsilon}) = \sqrt{\frac{6}{5}}$



$\epsilon = 0.1$

$\lambda_1(B^{f,\epsilon}) = \sqrt{\frac{2}{5}}$

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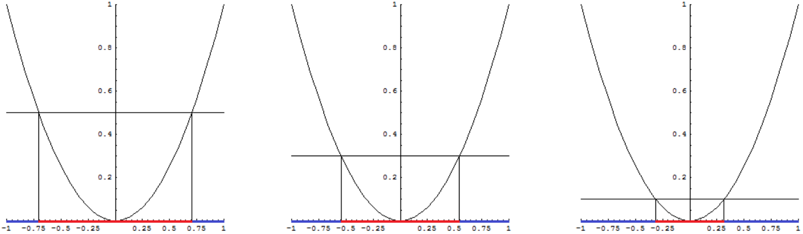

$\epsilon = 0.5$

$$\lambda_1(B^{f,\epsilon}) = \sqrt{2}$$

$$\sqrt{2} \leq 2\sqrt{2}\left(\frac{0.5}{1}\right)^{\frac{1}{2}} \cdot 2$$

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Basic Properties

- If f is (C, α)-good, it is also (C', α)-good for all $C' > C$ and (C, α')-good for all $\alpha' < \alpha$.

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- If f is (C, α)-good, then so is $g(\mathbf{x}) = f(k\mathbf{x})$ for all $k \in \mathbb{R}$.

Single-Variable Polynomials

Kleinbock and Margulis proved:

Theorem

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$$f(x) = \sum_{i=1}^{k+1} f(x_i) \prod_{j=1, j\neq i}^{k+1} \frac{x - x_j}{x_i - x_j}$$

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$$\implies \lambda_1(B^{f,\epsilon}) \leq 2k(k+1)^{\frac{1}{k}} \left(\frac{\epsilon}{\|f\|_B} \right)^{\frac{1}{k}} \lambda_1(B).$$

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Conjecture

All polynomial functions $f : \mathbb{R}^n \to \mathbb{R}$ of degree k are $(C, \frac{1}{k})$-good for some C.

Multivariable Linear Polynomials

Theorem

All linear polynomial functions $f : \mathbb{R}^n \to \mathbb{R}$ *are* $\left(\frac{4V_{n-1}}{V_n}, 1\right)$*-good.*[1]

[1] Here V_n stands for the volume of the unit ball in $\mathbb{R}^n$, i.e. $V_0 = 1, V_1 = 2, V_2 = \pi, V_3 = \frac{4\pi}{3}, \cdots$.

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- Then, $\|f\|_B = d(c + r)$ and the distance between the hyperplanes $f(\mathbf{x}) = \epsilon$ and $f(\mathbf{x}) = -\epsilon$ is $\frac{2\epsilon}{d}$.

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- Then, $\|f\|_B = d(c + r)$ and the distance between the hyperplanes $f(\mathbf{x}) = \epsilon$ and $f(\mathbf{x}) = -\epsilon$ is $\frac{2\epsilon}{d}$.
- We have four cases:

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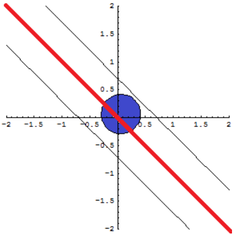
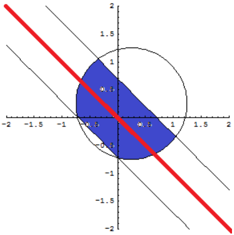
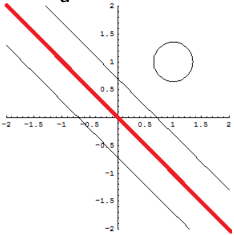


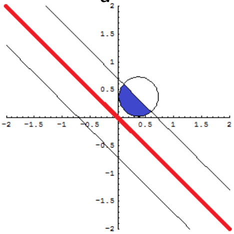
$\frac{\epsilon}{d} \geq r + c$



$r \geq c, \frac{\epsilon}{d} \leq r + c$



$r < c, \frac{\epsilon}{d} < c - r$



$r < c, c - r \leq \frac{\epsilon}{d} < c + r$

Multivariable Linear Polynomials

- Case 1: $B^{f,\epsilon} = B \implies$ Trivial
- Case 2: $B^{f,\epsilon}$ can be bounded by hypercylinder of height $\frac{2\epsilon}{d}$ and base $V_{n-1}r^{n-1} \implies \lambda_n(B^{f,\epsilon}) \leq \frac{4V_n}{V_{n-1}} \left(\frac{\epsilon}{\|f\|_B}\right)^{\alpha} \lambda_n(B)$
- Case 3: $B^{f,\epsilon} = \emptyset \implies$ Trivial
- Case 4: $B^{f,\epsilon}$ can be bounded by hypercylinder of height $\epsilon + r - c$ and base $V_{n-1}r^{n-1} \implies \lambda_n(B^{f,\epsilon}) \leq \frac{2V_n}{V_{n-1}} \left(\frac{\epsilon}{\|f\|_B}\right)^{\alpha} \lambda_n(B)$

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- This proof is similar to that of the linear polynomials, except we intersect two balls rather than a ball and the region between two hyperplanes.
- Interestingly, this case gives better C's than the optimal C for the entire family of quadratic polynomials, i.e. where the optimal C for single-variable quadratic polynomials is $2\sqrt{2}$ the optimal C for specifically this function ($f(x) = x^2$) is 2.

General Case

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All polynomial functions $f : \mathbb{R}^2 \to \mathbb{R}$ *of degree* k *are* $\left(\frac{8k}{\sqrt[k]{k!}}, \frac{1}{k}\right)$*-good.*

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Proof.

- Draw chords through the supremum of f on B, yielding single-variable polynomials. For each chord l, $\frac{\lambda_1(l \cap B^{f,\epsilon})}{\lambda_1(l \cap B)}$ is bounded $\left(\text{by } \frac{2k}{\sqrt[k]{k!}} \left(\frac{\epsilon}{\|f\|_B}\right)^{\frac{1}{k}}\right)$ so we can bound $\frac{\lambda_2(B^{f,\epsilon})}{\lambda_2(B)}$.

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- The problem reduces to how best to maximize such a region when $\frac{\lambda_1(l \cap B^{f,\epsilon})}{\lambda_1(l \cap B)}$ is a fixed p.

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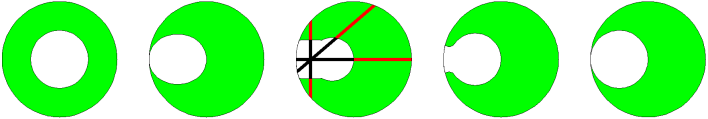
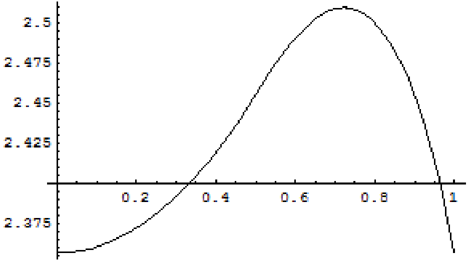
- The problem reduces to how best to maximize such a region when $\frac{\lambda_1(l \cap B^{f,\epsilon})}{\lambda_1(l \cap B)}$ is a fixed p.
- Letting c be the distance from the center of the circle to the supremum, we want to spread the points of the region as far away from the supremum as possible.

General Case



$c = 0$ $\quad c = \frac{1}{2}$ $\quad c = \frac{3}{4}$ $\quad c = \frac{9}{10}$ $\quad c = 1$



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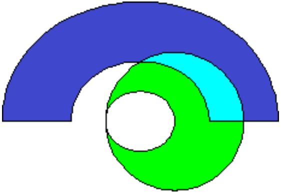

Lemma

Let R be a subset of circle S such that for every chord l of S through some point P, $\frac{\lambda_1(l \cap R)}{\lambda_1(l \cap S)}$ is at most p. Then $\frac{\lambda_2(R)}{\lambda_2(S)} < 4p - 2p^2$.

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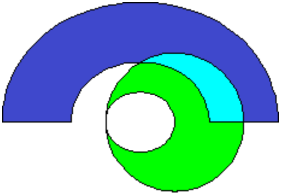

Lemma

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$$\lambda_2(B^{f,\epsilon}) < \left(4\frac{2k}{\sqrt[k]{k!}}\left(\frac{\epsilon}{\|f\|_B}\right)^{\frac{1}{k}} - 2\left(\frac{2k}{\sqrt[k]{k!}}\right)^2\left(\frac{\epsilon}{\|f\|_B}\right)^{\frac{2}{k}}\right)\lambda_2(B)$$

$$< \frac{8k}{\sqrt[k]{k!}}\left(\frac{\epsilon}{\|f\|_B}\right)^{\frac{1}{k}}\lambda_2(B).$$

Future Research

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- We would like to optimize our values for C. The estimations we used to get our values of C are clearly not optimal and we hope to lower our value of C.

Related Works

BPS S. Bacon, J. Pardo and G. Sturm, *(C,α)-good functions*, Brandeis University course project 2011.

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