Towards Generalizing Thrackles to Arbitrary Graphs

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Abstract

In the 1950s, John Conway came up with the notion of *thrackles*, graphs with embeddings in which no edge crosses itself, but every pair of distinct edges intersects each other exactly once. He conjectured that $|E(G)| \leq |V(G)|$ for any thrackle G, a question unsolved to this day. In this paper, we discuss some of the known properties of thrackles and contribute a few new ones.

Only a few sparse graphs can be thrackles, and so it is of interest to find an analogous notion that applies to denser graphs as well. In this paper we introduce a generalized version of thrackles called *near-thrackles*, and prove some of their properties. We also discuss a large number of conjectures about them which seem very obvious but nonetheless are hard to prove. In the final section, we introduce *thrackleability*, a number between 0 and 1 that turns out to be an accurate measure of how far away a graph is from being a thrackle.

1 Introduction

In 1952, John Conway came up with the notion of thrackles and formulated his famous Thrackle Conjecture, which remains one of the most difficult open problems in combinatorics. One of the reasons why this is of interest is the apparent simplicity of these objects, and the beautiful known mathematics surrounding them. Another reason is the fact that — its seeming simplicity notwithstanding — it appears to be an extremely challenging problem, since it combines arguments from graph theory with arguments from topology and geometry. Throughout this paper, we will only consider graphs without loops and multiple edges. Conway conjectured that a *thrackleable graph*, a graph in which no edge crosses itself but every pair of edges intersects each other precisely once, has at most as many edges as vertices.

The conjecture is still open, but better and better bounds have been proven over the years. It is known that $|E(G)| \leq c|V(G)|$ for some constant c, which implies that thrackles are *sparse* graphs. The constant c has also gotten closer and closer to 1 over the years. Lovász, Pach and Szegedy [4] proved the result for $c \approx 3$, and Cairns and Nikolayevsky [1] proved it for $c \approx 1.5$. Today the best-known bound is $c \approx 1.428$, due to Fulek and Pach [3]. The conjecture is known to be true for some subclasses of thrackles, such as for *linear thrackles*, which are graphs that have thrackle drawings that use only straight lines.

This paper is arranged as follows. In Section 2, we will briefly set up the preliminaries and notation that we will use throughout the rest of the paper. Then in Section 3, we will briefly discuss some basic properties of thrackles and formulate a new theorem related to their chromatic number.

Theorem 1.1. *A thrackle G has chromatic number at most* 3.

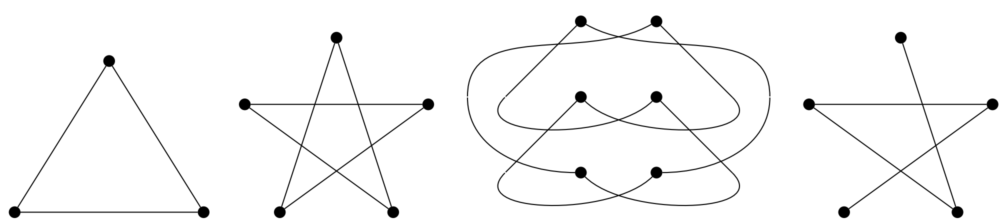
In Section 4 we will switch gears and talk about the problem of generalizing thrackles. Thrackles are interesting objects, but the problem is that they only apply to sparse graphs, since the number of edges is only linear in the number of vertices. To fix this, we wish to find some embedding for any graph G that is in some sense "near" a thrackle drawing of G. This is the motivation behind the definition of *near-thrackle drawings*. A near-thrackle drawing will be an embedding of a graph that will first maximize the number of pairs of edges that intersect once, and then maximize the number of pairs of edges that do not cross, and then the ones that cross twice, then thrice, and so on. In particular, if G is a thrackle, its near-thrackle drawing should just be a thrackle drawing of G. In Section 5, we will consider the theory of near-thrackle drawings in the case when we only allow straight lines, forming *linear near-thrackles*. We will finally turn to asymptotics in Section 6, and then discuss the notion of *thrackleability* and pose some of the interesting questions it raises.

2 Preliminaries and Notation

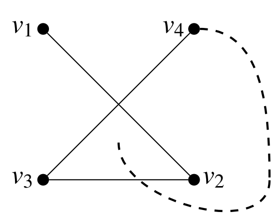
We start with some basic definitions.

Definition 2.1. *A* ***thrackle drawing*** *is a graph embedding where no edge crosses itself, but every pair of distinct edges intersects each other* ***exactly*** *once; this point of intersection is allowed to be a common endpoint, but cannot be tangential between the two edges. A* ***thrackle*** *is a graph that admits a thrackle drawing.*

For instance, the following graphs (the *cycle graphs* C_3, C_5, C_6 and the *path graph* P_4) are all thrackles because of the way we have drawn them.



However, note that the 4-cycle C_4 is *not* a thrackle.



If $v_1v_2v_3v_4v_1$ had been a thrackle, we would be able to start from v_1 and trace out the cycle and get to v_4 without violating the thrackle property. But then note that v_4v_1 would have to intersect v_2v_3, and then we would end up in the small triangle as in the figure above, and there would be no way to escape and connect back to v_1 without creating superfluous crossings.

In fact, it is known (and we will prove shortly) that C_n for $n \geq 3$ is a thrackle except when $n = 4$. Note that it is obvious that if G is a thrackle, then so is any subgraph G' of G, since we can just start with a thrackle drawing of G and then delete edges and vertices from that same drawing to get a thrackle drawing of $G' \subseteq G$.

3 Properties of Thrackles

3.1 Known Results

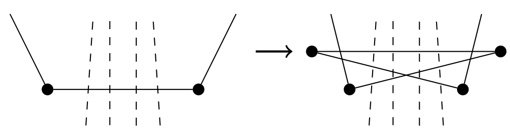
The following results are classic and can be found readily in literature. (See, for instance, [4, 5, 6])

Theorem 3.1. *If G is a **linear** thrackle (constrained to be drawn only using straight lines), then* $|E(G)| \leq |V(G)|$.

The proof of this theorem is due to Pach and Sterling [5]. We now prove the result mentioned earlier about almost all cycles being thrackles. It is clear that C_3 is a thrackle. We can also easily verify that C_6 also has a thrackle drawing. The following claim, due to Wehner [6], then gives us a two-step inductive proof that every C_k is a thrackle for $k \in \mathbb{N} \setminus \{1, 2, 4\}$.

Claim 3.1. *If C_n is a thrackle, then C_{n+2} is also a thrackle.*

Proof. If we replace any path of length 3 to a path of length 5 as follows, each pair of edges will still intersect exactly once, and so we are left with a thrackle.



□

In fact, Lovász, Pach and Szegedy [4] proved the following structural property of thrackles. We omit the proof since it is too technical, though in essence it just involves comparing parities in two different ways.

Claim 3.2. *A thrackle cannot contain two vertex-disjoint odd cycles.*

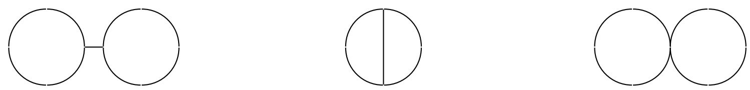
Let us restate an easy claim mentioned once before as a proposition, since it makes it easier to motivate an approach to Conway's conjecture. The proof is trivial.

Proposition 3.1. *If G is a thrackle, then any subgraph G' of G is also a thrackle.*

Corollary 3.1.1. *If G has a C_4-subgraph, G is not a thrackle.*

Using these tools, one can carefully prove the following claim. The details are left as an exercise for the enterprising reader.

Claim 3.3. *If Conway's Conjecture is false, then a* ***minimal counterexample*** *will be topologically homeomorphic to one of the following three shapes, as shown in [6].*

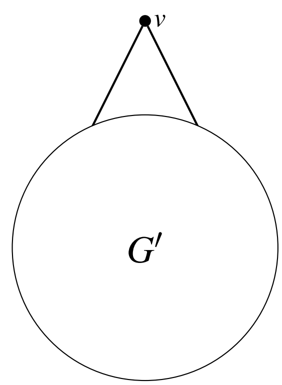


Furthermore, Rubinstein (unpublished) showed that if any one of these counterexamples exists, then so do the other two.

3.2 Chromatic Numbers of Thrackles

In this subsection, we will prove Theorem 1.1. Recall that the theorem states that every thrackle is 3-colorable. Note that this is not a very obvious result in spite of the fact that thrackles do not have C_4-subgraphs and hence K_4-subgraphs. We might expect that if a graph does not contain small cycles, then it looks locally like a tree, and is therefore 2-colorable. However, this is false, since graphs of arbitrarily large girth and chromatic number are known, a deep result due to Erdős [2].

Proof of Theorem 1.1. We use induction on the number of vertices n. Clearly for $n \in \{1,2,3\}$, the result is trivial. Suppose now for $n \geq 4$ we have some thrackle G on n vertices. First we claim that G has a vertex of degree at most 2. If not, then all vertices in G have degree at least 3, and so summing the degrees we get $2|E(G)| = \sum_{v \in V(G)} \deg(v) \geq 3|V(G)|$, so that $|E(G)| \geq 1.5|V(G)|$, which contradicts the known bound of $|E(G)| \leq 1.428|V(G)| < 1.5|V(G)|$ due to Fulek and Pach [3]. So assume $v \in V(G)$ has degree at most 2.



G

Consider the graph $G' = G \setminus \{v\}$. Clearly G' is a subgraph of G and is therefore a thrackle. By the induction hypothesis, G' is 3-colorable. Now when we add v back in to form G from G', we can extend a proper 3-coloring of G' to a proper 3-coloring of G, since the neighbors of v in G use up at most two of our three available colors. Hence G is 3-colorable as well, completing the proof. $\square$

4 Near-Thrackle Drawings

4.1 Definition and examples

Definition 4.1. *For any graph G, a* ***near-thrackle drawing*** *of G is an embedding of G in the plane satisfying the following conditions:*

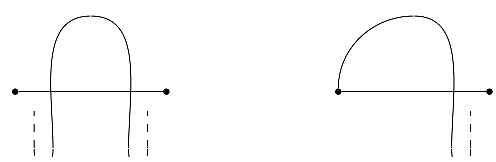
- *First, out of all the embeddings of G, choose only the ones that maximize the number of pairs of edges that cross exactly once.*
- *Then, out of these embeddings of G, choose only the ones that maximize the number of pairs of edges that do not cross.*
- *Iterate the process by maximizing the number of pairs of edges that cross* $2,3,4,\dots$ *times.*

Note that if G is a thrackle, this algorithm stops after the first step. We have the following conjecture that seems true based on an extensive search.

Conjecture 4.1. *For any input graph G, the algorithm to determine a near-thrackle drawing (Definition 4.1) stops after the first two steps. That is, for every graph G, there exists a (not necessarily unique) drawing that maximizes the* 1*-crossings and only uses* 0*- and* 1*-crossings.*

The reason we expect this conjecture to be true is the following intuition. Suppose for a graph G, a near-thrackle drawing has m_1 pairs of edges that cross once, m_0 pairs of edges that do not cross, and $m_{\geq 2}$ pairs that cross twice or more, for $m_{\geq 2} \geq 1$. Then, the total number of pairs of edges of G, which is constant for any given G, is $m_1 + m_0 + m_{\geq 2}$. We already know there exists an embedding of G. In that embedding, suppose there are m'_1 pairs of edges that cross once and m'_0 pairs that do not cross. Then, $m'_1 + m'_0 = m_1 + m_0 + m_{\geq 2} > m_1 + m_0$. Since we started by maximizing m_1, we have $m_1 \geq m'_1$, so that $m'_0 < m_0$, which appears to be an extremely hard example to construct, though not entirely impossible. An interesting question would be to figure out what such a drawing would look like, if it exists, and what, if anything, it would tell us about the underlying topology.

If this conjecture were false, let us see what kind of configurations we might expect in a counterexample.



Clearly, if $m_{\geq 2} \geq 1$, then one of the above configurations must appear in our graph drawing (since there must be a pair of edges intersecting more than once). However, if the configuration above appears in our drawing with nothing in the bounded region, then we can get a better drawing just by "pulling" the two edges apart, like a type II Reidemeister move. Therefore, the bounded region must contain a vertex of degree at least 2. This fact is straightforward to prove. We can prove a small number of other statements about the bounded region, but not enough to get a counterexample or a contradiction.

The following is an example of a graph G and a near-thrackle drawing. In this near-thrackle drawing, it is easy to check that there are 27 pairs of edges crossing each other exactly once, and one pair of edges which do not cross, namely $\overline{a_1a_2}$ and $\overline{ac}$.

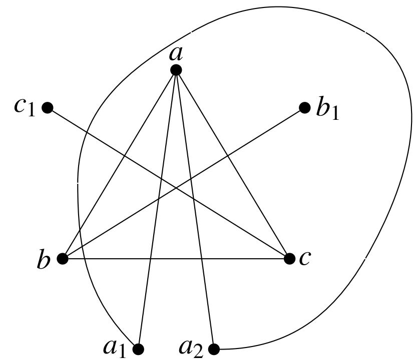
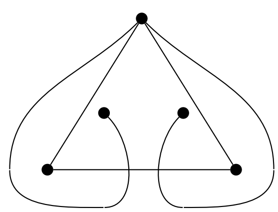


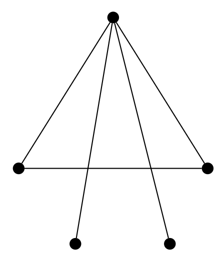
Figure 1: An example of a near-thrackle drawing of a graph G

Claim 4.1. *Figure 1 does indeed depict a near-thrackle drawing of the graph G.*

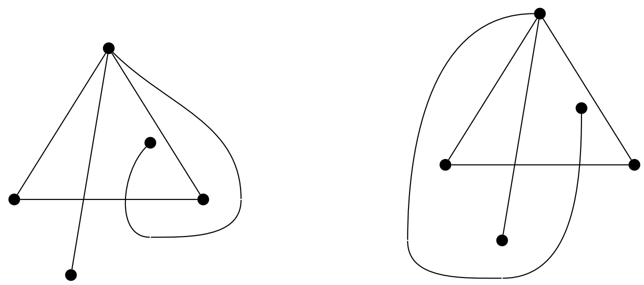
Proof. It suffices to prove that G is not thrackleable. This can be shown by arguing that the wedge of two triangles (which is a subgraph of G) is not a thrackle.

Since a triangle has a unique thrackle embedding, we can without loss of generality fix one of them. Now consider what happens when we add the two edges adjacent to the "wedge" vertex. It is easy to see that there are essentially four ways to add this pair of edges and still maintain the thrackle property. In one of them, the two degree-1 vertices lie inside the original triangle; in another, they both lie outside; and in two other ways they lie in different regions.





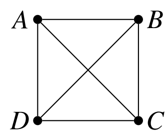
In these two drawings, we have to connect the two degree-1 vertices by an edge, which should intersect each side of the original triangle exactly once (to maintain the thrackle property). But clearly, by parity counting, this edge will then end up on a different region away from where we want it to connect. The two remaining cases are the following.



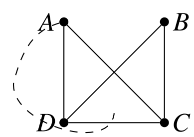
We can no longer use a parity argument, but by explicitly starting from one of the degree-1 vertices, we can argue topologically (using the Jordan curve theorem implicitly) that we cannot complete the drawing in any way. We leave it to the reader to verify the details.

This implies that G is not thrackleable, and we are done. □

Finding a near-thrackle drawing of K_4 is similar.

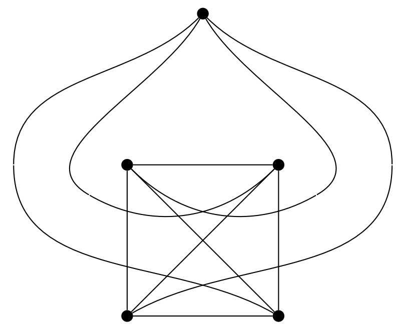


If a graph contains C_4, it cannot be a thrackle. The usual (non-planar) way of drawing K_4 turns out to be a near-thrackle drawing. In this particular drawing, we can see 13 pairs of 1-crossing and 2 pairs of 0-crossing edges. Since we know that K_4 is not a thrackle and hence there cannot be 15 pairs of 1-crossing edges, our goal at best would be to find 14 1-crossing pairs of edges with either a 0-crossing or an n-crossing (for some $n \geq 2$). In the drawing above, the two 0-crossing pairs of edges are $\overline{AB}$ and $\overline{CD}$, and $\overline{BC}$ and $\overline{DA}$, so let's try to make 14 pairs of 1-crossing edges by crossing $\overline{AB}$ and $\overline{CD}$.



However, when we cross $\overline{AB}$ and $\overline{CD}$, it is unavoidable to cross one of already crossed lines such as $\overline{AC}$, $\overline{BD}$, $\overline{AD}$, $\overline{BC}$, $\overline{CD}$. As a result, it gives a strictly worse embedding than the standard drawing of K_4 itself. So this standard drawing is the near-thrackle embedding. This is not a rigorous proof by any means, but we leave it to the reader to check that it can be made into one easily.

Finding a near-thrackle embedding of K_5 turns out to be much harder. So far, this seems to be the best embedding.



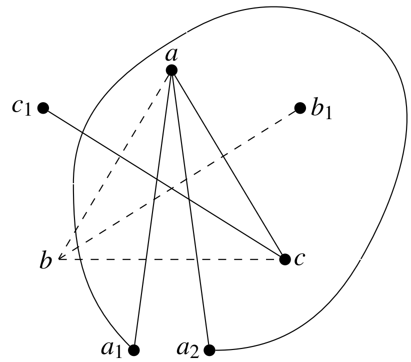
Here 37 pairs of edges intersect once, 6 pairs do not intersect, and 2 pairs of edges intersect twice. If this embedding is a near-thrackle drawing, Conjecture 4.1 ends up false. It can be proven with some additional work that there can be at most 39 pairs of edges intersecting exactly once in any drawing of K_5.

4.2 Strong and weak deletion conjectures

Since near-thrackle drawings are generalized forms of thrackles, we expect them to have some characteristics of thrackles. It seemed at first glance that the so-called **strong deletion conjecture** would be true. This was due to the following: Suppose we have a near-thrackle drawing of a graph G. Pick any $e \in E(G)$, and delete e from that drawing. Then this drawing would be expected to be a near-thrackle drawing of $G \setminus \{e\}$. However, this conjecture turns out to be false, as evidenced by the following counterexample. Note that this is a counterexample to both the strong deletion conjecture for edges, as well as to the one for *vertices*, which states that given a near-thrackle drawing, we can delete any vertex to obtain a near-thrackle drawing for the corresponding subgraph. In the following figure, there are two ways to delete an edge, and since all edges in K_4 are equivalent, the resulting subgraph should be the same.

Out of 10 pairs of edges, the graph on the left would have 9 pairs of 1-crossing and 1 pair of 0-crossing edges. However, the graph on the right would have 8 pairs of 1-crossing and 2 pairs of 0-crossing edges. Thus the strong edge deletion conjecture is false.

To disprove the strong vertex deletion conjecture, we can use Figure 1. Deleting vertex b leaves a non-thrackle drawing of a thrackleable graph, and hence the conjecture is false.



The disproof follows because the graph above has a thrackle embedding.

This leads to the so-called **weak deletion conjectures**, which has been found true for all studied examples so far.

Conjecture 4.2 (Weak deletion for edges)**.** *Suppose we have a near-thrackle drawing of a graph G. Then there exists some $e \in E(G)$ such that deleting e from this drawing yields a near-thrackle drawing of $G \setminus \{e\}$.*

Conjecture 4.3 (Weak deletion for vertices)**.** *Suppose we have a near-thrackle drawing of a graph G. Then there exists some $v \in V(G)$ such that deleting v from this drawing yields a near-thrackle drawing of $G \setminus \{v\}$.*

Note that the conjecture for edges does *not* imply the one for vertices in case of weak deletion.

5 Linear Near-Thrackles

5.1 Definition and examples

Definition 5.1. *For any graph G, a* ***linear near-thrackle drawing*** *of G is a near-thrackle embedding of G subject to the constraint that all edges must be drawn as straight lines.*

Note that any two non-overlapping straight lines can intersect at most once. So the edges of a linear near-thrackle drawing intersect each other at most once. So unlike general near-thrackle drawings, it is clear that the algorithm to determine a linear near-thrackle drawing (Definition 5.1) actually does stop after the second step. An example would be Figure 2.

The following theorem follows because of the well-known fact that the number of crossings is maximized when points are placed in a convex position. For a convex n-gon, the number of interior crossings is $\binom{n}{4}$, since any interior crossing is formed by two diagonals which are defined uniquely by the quadrilateral formed by choosing four of the n vertices.

Theorem 5.1. *A linear near-thrackle drawing of K_n is obtained by taking the n vertices in convex position, and then drawing all possible edges between them.*

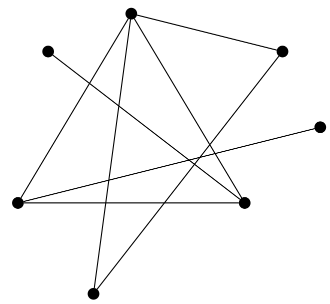


Figure 2: An example of a linear near-thrackle.

In general, this seems to be the unique linear near-thrackle drawing of K_n. A nice consequence of the theorem about complete graphs above is that linear near-thrackle drawings of K_n have $\binom{n}{4} + n\binom{n-1}{2} = \frac{n(n-1)(n-2)(n+9)}{24}$ pairs of edges that cross exactly once, and the remaining pairs do not cross at all.

The same result seems to be true for complete bipartite graphs as well. It is verifiable for $K_{2,3}$ and $K_{3,3}$.

Conjecture 5.1. *A linear near-thrackle drawing of $K_{m,n}$ is obtained by taking $m+n$ vertices in convex position, and then defining m contiguous ones as one side of the partition, the n others as the other side of the partition, and drawing all possible edges between them.*

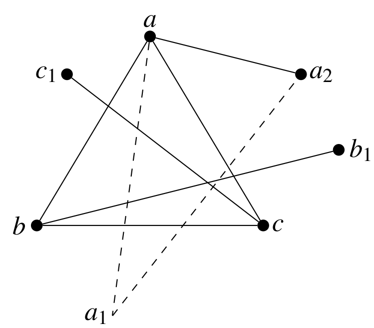
If the conjecture is true, a similar expression can be obtained for $K_{m,n}$. It can be easily checked that in this case, linear near-thrackle drawings of $K_{m,n}$ have $m\binom{n}{2} + n\binom{m}{2} + \binom{n}{2}\binom{m}{2}$ pairs of edges that cross exactly once.

5.2 Strong and weak deletion conjectures for linear near-thrackles

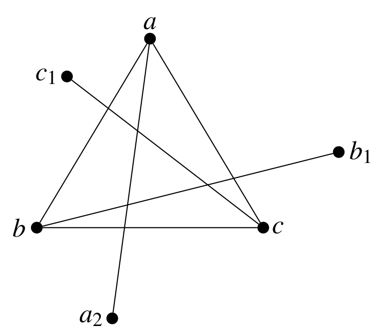
We can formulate the deletion conjectures exactly as for general near-thrackles. Note that these are different questions, given the additional constraints in drawing linear near-thrackles.

However, since the near-thrackle drawing and the linear near-thrackle drawing are the same for K_4, the strong edge deletion conjecture for linear near-thrackle also turns out to be false, by the same counterexample.

For strong vertex deletion conjecture for linear near-thrackle, we will use Figure 2 to prove that the strong vertex deletion conjecture is false. To prove that, we will delete vertex a_1 from Figure 2.



To finish off the argument, note that the graph above has a strictly better drawing by shifting the vertex a_2, so this cannot be a near-thrackle drawing.



We once again wrap up this section by stating the weak deletion conjecture, this time for linear near-thrackles.

Conjecture 5.2 (Weak deletion for edges, linear case)**.** *Suppose we have a linear near-thrackle drawing of a graph G. Then there exists some $e \in E(G)$ such that deleting e from this drawing yields a linear near-thrackle drawing of $G \setminus \{e\}$.*

Conjecture 5.3 (Weak deletion for vertices, linear case)**.** *Suppose we have a linear near-thrackle drawing of a graph G. Then there exists some $v \in V(G)$ such that deleting v from this drawing yields a linear near-thrackle drawing of $G \setminus \{v\}$.*

It is worth observing that the conjectures for the general case do not, in fact, imply the ones for the linear case. We leave it to the reader to verify this straightforward fact.

6 Thrackleability

Because of the fact that, given a graph G, there is no easy way to determine m_1, the number of pairs of edges from $E(G)$, that cross each other exactly once in a near-thrackle drawing of G (or in other words, the maximum number of pairs of edges that cross each other exactly once over all drawings of G on the plane), we can consider for any graph the ratio $m_1/\binom{|E(G)|}{2}$ and use it as a measure of how "close" G is to being a thrackle, in some sense. This yields the following definition.

Definition 6.1. *The **thrackleability** $\varphi(G)$ of a graph G with at least two edges is defined as the quantity*

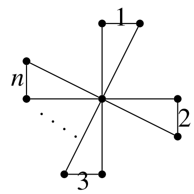
$$\frac{m_1}{\binom{|E(G)|}{2}},$$

where m_1 is the number of pairs of edges in $E(G)$ that cross each other exactly once in any near-thrackle drawing of G.

Note that $\varphi(G) = 1$ if and only if G is a thrackle. Furthermore, note that $\varphi(G) \neq 0$ for any G, since any graph with two edges has a drawing in which $m_1 \geq 1$.

It is easy to compute the thrackleability of the graphs we have used throughout this paper as examples. An interesting question is what happens in the asymptotics limit. Stated precisely, if $\mathscr{F}$ is a family of graphs of increasing size, then what are the families $\mathscr{F} = (G_1, G_2, \ldots)$ for which $\varphi(G_n)$ converges to a finite value as $n \to \infty$? If this happens, we denote this value by $\varphi(\mathscr{F})$.

For an example, suppose $\mathscr{F} = (G_1, G_2, \ldots)$ is the family of triangle-wedges, where G_n is the wedge of n triangles. What can we say about $\varphi(\mathscr{F})$?



Proposition 6.1. *If $\mathscr{F}$ is the family of triangle-wedges, then $\varphi(\mathscr{F})$ exists and is bounded above by* $8/9$.

Proof. We know by our earlier arguments that G_2 is not a thrackle. So in every copy of G_2 in G_n for $n \geq 2$, there is a pair of edges that do not cross each other exactly once. In particular, note that for any two triangles in G_n, we have a unique copy of G_2, and hence a pair of edges corresponding only to that copy of G_2 that do not intersect exactly once. So, there are at least $\binom{n}{2}$ pairs of edges in G_n that do not intersect each other, and so

$$\varphi(G_n) \leq 1 - \frac{\binom{n}{2}}{\binom{3n}{2}}.$$

In the limit $n \to \infty$, therefore, we get

$$\varphi(\mathscr{F}) \leq 1 - \lim_{n\to\infty} \frac{\binom{n}{2}}{\binom{3n}{2}} = 1 - \frac{n^2}{9n^2} = \frac{8}{9}. \qquad \square$$

We can even ask the same questions for linear near-thrackles.

Definition 6.2. *The **linear thrackleability** $\varphi^+(G)$ to be $m_1'/\binom{|E(G)|}{2}$, where m_1' is the number of pairs of edges that intersect each other exactly once in a linear near-thrackle drawing of G.*

As before, $0 < \varphi^+(G) \leq 1$ for any G, with $\varphi^+(G) = 1$ if and only if G is a linear thrackle, for instance, any odd cycle.

We can ask the same bound-related questions about $\varphi^+(G)$. For instance, we have the following proposition.

Proposition 6.2. *Let $\mathscr{G}$ be the family of complete graphs $\{K_n\}_{n\in\mathbb{N}}$. Then, $\varphi^+(\mathscr{G}) = 1/3$, where φ^+ for a family of graphs is defined as for φ.*

Proof. From the discussions following Theorem 5.1, we get

$$\varphi^+(\mathscr{G}) = \lim_{n\to\infty} \frac{\frac{n(n-1)(n-2)(n+9)}{24}}{\binom{\binom{n}{2}}{2}} = \frac{8}{24} = \frac{1}{3}. \qquad \square$$

Proposition 6.3. *Let $\mathscr{G}$ be the family of complete graphs $\{K_n\}_{n\in\mathbb{N}}$, and $\mathscr{G}\backslash e$ be the family $\{K_n \setminus e\}_{n\in\mathbb{N}}$, the family of complete graphs with precisely one edge deleted from each. Then, $\varphi^+(\mathscr{G}\backslash e) = 1/3$, where φ^+ for a family of graphs is defined as the obvious restriction of φ to linear drawings.*

Proof. From the discussions following Theorem 5.1, we get

$$\varphi^+(\mathscr{G}) \leq \lim_{n\to\infty} \frac{\frac{n(n-1)(n-2)(n+9)}{24}}{\binom{\binom{n}{2}}{2}} = \frac{8}{24} = \frac{1}{3}.$$

$$\varphi^+(e) \geq 2(n-2).$$

$$\varphi^+(\mathscr{G}\backslash e) \leq \lim_{n\to\infty} \frac{\frac{n(n-1)(n-2)(n+9)}{24} - 2(n-2)}{\binom{\binom{n}{2}-1}{2}} = \frac{8}{24} = \frac{1}{3}. \qquad \square$$

Proposition 6.4. *Let $\mathscr{G}$ be the family of complete graphs $\{K_n\}_{n\in\mathbb{N}}$, and $\mathscr{G}\backslash te$ be the family of complete graphs$\{K_n\}_{n\in\mathbb{N}}$ with t disjoint edges deleted where $t \leq \frac{\lfloor n \rfloor}{2}$ Then, $\varphi^+(\mathscr{G}\backslash te) = 1/3$, where φ^+ for a family of graphs is defined as for φ.*

Proof. Starting from any drawing of K_n, any edge we delete gets read of at least $2(n-2)$ crossings, corresponding to its degree in the rest of the graph. So after deleting t disjoint edges, we have destroyed at least $2t(n-2)$ 1-crossings. Since the number of 1-crossings in a linear near-thrackle drawing of K_n is known from Theorem 5.1, we get an upperbound to $\varphi^+(\mathscr{G}\backslash te)$ to be

$$\varphi^+(\mathscr{G}\backslash te) \leq \lim_{n\to\infty} \frac{\frac{n(n-1)(n-2)(n+9)}{24} - 2t(n-2)}{\binom{\binom{n}{2}-1}{2}} = \lim_{n\to\infty} \frac{\frac{n(n-1)(n-2)(n+9)}{24} - 2\frac{\lfloor n \rfloor}{2}(n-2)}{\binom{\binom{n}{2}-1}{2}} = \frac{8}{24} = \frac{1}{3}.$$

The fact that this bound can be attained can be seen by drawing a usual linear near-thrackle drawing of K_n, picking t disjoint edges along with the outer boundary of this drawing and deleting them. $\square$

Conjecture 6.1. *Let $\mathscr{B}$ be the family of complete bipartite graphs $\{K_{m,n}\}_{m,n\in\mathbb{N}}$. Then, $\varphi^+(\mathscr{B}) = 1/2$, where φ^+ for a family of graphs is defined as for φ.*

The reason why we expect this to be true is the following. If the Conjecture 5.1 is true, we get

$$\varphi^+(\mathscr{B}) = \lim_{m,n\to\infty} \frac{m\binom{n}{2} + n\binom{m}{2} + \binom{n}{2}\binom{m}{2}}{\binom{m+n}{2}} = \frac{1}{2}.$$

This raises an obvious question. How close to 0 can $\varphi(G)$ or $\varphi^+(G)$ get? Are there specific families of graphs for which asymptotically φ or φ^+ gets arbitrarily close to 0? If not, what is a good lower bound for φ or φ^+? Can we get $\varphi(G)$ to be at most $1/2$?

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References

[1] G. Cairns and Y. Nikolayevsky, *Bounds for generalized thrackles*, Discrete and Computational Geometry, **23**:2 (2000), 191-206.

[2] P. Erdős, *Graph Theory and Probability*, Canadian Journal of Mathematics **11** (1959), 34-38.

[3] R. Fulek and J. Pach, *A computational approach to Conway's thrackle conjecture*, Computational Geometry **44**:6-7 (2011), 345-355.

[4] L. Lovász, J. Pach and M. Szegedy, *On Conway's thrackle conjecture*, Discrete and Computational Geometry **18**:4 (1997), 369-376.

[5] J. Pach and E. Sterling, *Conway's conjecture for monotone thrackles*, American Mathematical Monthly **118**:6 (2011), 544-548.

[6] S. Wehner, *On the thrackle problem*, `http://www.thrackle.org/thrackle.html`, accessed 9/25/2013.