Infinitesimal Cherednik Algebras

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Second Annual MIT PRIMES Conference, May 19, 2012

Introduction

- Motivation: Generalization of the basic representation theory of different algebras, including $\mathfrak{U}\mathfrak{sl}_n$ and $\mathfrak{U}\mathfrak{sp}_{2n} \ltimes A_n$.
- Main objects: Infinitesimal Cherednik algebras $H_\alpha(\mathfrak{gl}_n)$ and $H_\alpha(\mathfrak{sp}_{2n})$.
- Work from last year:
  1. Computation of entire center for the case $H_\alpha(\mathfrak{gl}_2)$.
  2. Computation of the Shapovalov determinant.
  3. Classification of finite-dimensional irreducible representations.
- New results:
  1. Proof of the formula for the first central element of $H_\alpha(\mathfrak{gl}_n)$.
  2. Conjecture regarding the entire center for all $H_\alpha(\mathfrak{gl}_n)$.
  3. Computation of Poisson center for $H_\alpha(\mathfrak{gl}_n)$ and $H_\alpha(\mathfrak{sp}_{2n})$.

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Definition of Infinitesimal Cherednik Algebras

Definition

Let V be the standard n-dimensional column representation of $\mathfrak{gl}_n$, and V^* be the row representation, $\alpha : V \times V^* \to \mathfrak{Ugl}_n$.
The **infinitesimal Cherednik algebra** H_α is defined as the quotient of $\mathfrak{Ugl}_n \ltimes T(V \oplus V^*)$ by the relations:

$$[y, x] = \alpha(y, x),\ [x, x'] = [y, y'] = 0$$

for all $x, x' \in V^*$ and $y, y' \in V$. In addition, H_α must satisfy the PBW property.

Acceptable deformations α

- Etingof, Gan, and Ginzburg proved that α is given by $\sum_{j=0}^{k} \alpha_j r_j$ where r_j is the coefficient of z^j in the expansion of

$$\operatorname{tr}(x(1 - zA)^{-1}y)\det(1 - zA)^{-1}$$

- We can naturally consider the α_j as coefficients of a polynomial $\alpha(z) = \sum \alpha_j z^j$.
- If $\alpha(z)$ is a linear polynomial, $H_\alpha \cong U(\mathfrak{sl}_{n+1})$.

Center is Polynomial Algebra

- Let t_1, t_2, ..., t_n be the generators for the center of H_0:

$$t_i = \sum_{j=1}^{n} x_j[\beta_i, y_j],$$

  where β_i is defined by $\sum_{i=0}^{n}(-1)^i\beta_i z^i = \det(1 - zA)$.
- Tikaradze proved that there exist unique (up to constant) $c_1, c_2, \ldots, c_n \in \mathfrak{z}(\mathfrak{U}\mathfrak{g})$ such that $\mathfrak{z}(H_\alpha) = k[\underbrace{t_1 + c_1}_{t_1'}, \underbrace{t_2 + c_2}_{t_2'}, \ldots, \underbrace{t_n + c_n}_{t_n'}]$

Poisson infinitesimal Cherednik algebra

- A Poisson algebra is a commutative algebra with a *Poisson bracket* that satisfies

$$\{ab, c\} = b\{a, c\} + a\{b, c\}.$$

Note that the Lie bracket satisfies [ab,c]=a[b,c]+[a,c]b.

- We can study the Poisson analogue of infinitesimal Cherednik algebras, defined as $S(\mathfrak{gl}_n) \ltimes S(V \oplus V^*)$ with $\{a, b\} = [a, b]$ for $a, b \in \mathfrak{gl}_n \ltimes (V \oplus V^*)$.
- The Poisson center consists of elements c that satisfy $\{c, a\} = 0$ for all a.
- The Poisson bracket approximates its corresponding Lie bracket. **Our Goal: to obtain information about the Lie bracket from the Poisson bracket.**

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Computation of Poisson Center

1. Let $g \in \mathfrak{U}\mathfrak{gl}_n$. Then, $\{g, y\} = \sum_{i,j=1}^n \frac{\partial g}{\partial e_{ij}} \{e_{ij}, y\}$.
2. $$\{t_k, y\} = \sum_{j=1}^n \left(\mathrm{Res}_{z=0}\, \alpha(z^{-1}) \frac{\mathrm{tr}(x_j(1-zA)^{-1}y)}{z \det(1-zA)} dz \right) \{\beta_k, y_j\}.$$
3. Thus, if $\{t_k + c_k, y\} = 0$,
$$\sum_{i,j=1}^n \frac{\partial c_k}{\partial e_{ij}} \{e_{ij}, y\} = -\sum_{j=1}^n \left(\mathrm{Res}_{z=0}\, \alpha(z^{-1}) \frac{\mathrm{tr}(x_j(1-zA)^{-1}y)}{z \det(1-zA)} dz \right) \{\beta_k, y_j\}.$$
4. **Key idea: since all terms above are GL_n-invariant and diagonalizable matrices are dense in $\mathfrak{gl}_n$, we can assume A is diagonal.**

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4. **Key idea: since all terms above are GL_n-invariant and diagonalizable matrices are dense in $\mathfrak{gl}_n$, we can assume A is diagonal.**

The Poisson Center

Definition

Let $c(t)$ be the generating function of c_i: $c(t) = \sum_{i=1}^{n} (-t)^i c_i$.

Theorem

$$c(t) = -\operatorname{Res}_{z=0}\, \alpha(z^{-1}) \frac{\det(1-tA)}{\det(1-zA)} \frac{tz^{-2}}{1-tz^{-1}} dz.$$

Examples

Case c_1 :

$$c_1 = \mathrm{Res}_{z=0}\,\alpha(z^{-1})\,\det(1 - zA)^{-1} z^{-2} dz = \sum_i \alpha_i \,\mathrm{tr}\, S^{i+1} A.$$

Case c_2 :

$$\begin{aligned} c_2 &= \mathrm{Res}_{z=0}\,\alpha(z^{-1})\,\det(1 - zA)^{-1} z^{-2} \left(\mathrm{tr}\, A - z^{-1}\right) dz \\ &= \sum_i \alpha_i \left(\beta_1 \,\mathrm{tr}\, S^{i+1} A - \mathrm{tr}\, S^{i+2} A\right). \end{aligned}$$

Poisson to non-Poisson

Theorem

$$[\beta_k, y] = \left\{ \sum_{i=0}^{k-1} \binom{k-n}{i+1} \frac{1}{k-n} \beta_{k-i}, y \right\}.$$

Theorem

$$[\operatorname{tr} S^k A, y] = \left\{ \sum_{i=0}^{k-1} \frac{1}{k+n+1} \binom{k+n+1}{i} (-1)^i \operatorname{tr} S^{k-i} A, y \right\}.$$

Change of basis

- Let $f(z)$ be the polynomial satisfying $f(z) - f(z-1) = \partial^n(z^n\alpha(z))$, and $g(z) = z^{1-n}\frac{1}{\partial^{n-1}}f(z)$.
- Note that if $g(z) = z^{k+1}$,

$$\alpha(z) = \sum_{i=0}^{k-1} \frac{1}{k+n+1}\binom{k+n+1}{i}(-1)^i z^{k-i}.$$

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$$\alpha(z) = \sum_{i=0}^{k-1} \frac{1}{k+n+1}\binom{k+n+1}{i}(-1)^i z^{k-i}.$$

- Thus, $[\sum g_{j+1} \operatorname{tr} S^{j+1}A, y] = \{\sum \alpha_j \operatorname{tr} S^{j+1}A, y\}$.

Change of basis

- Conclusion:

$$[t_1, y] = \sum_{i=1}^{n} [x_i, y] y_i = \sum_{i=1}^{n} \{x_i, y\} y_i$$
$$= -\{\mathrm{Res}_{z=0}\, \alpha(z^{-1}) \det(1 - zA)^{-1} z^{-2} dz, y\}$$
$$= -[\mathrm{Res}_{z=0}\, g(z^{-1}) \det(1 - zA)^{-1} z^{-1} dz, y].$$

Hence, $c_1 = \mathrm{Res}_{z=0}\, g(z^{-1}) \det(1 - zA)^{-1} z^{-1} dz$.

The General Formula for the Central Element

Conjecture

Let $c(t)$ be the generating function of the c_i, i.e., $c(t) = \sum_{i=1}^n t^i c_i$. Let

$$h(t,z) = z^{1-n}\left(\frac{2\sinh\frac{\partial}{2}}{\partial}\right)^{n-1}\frac{1}{1+tz}\left(\frac{1}{2\sinh\frac{\partial}{2}}\right)^{n-1} f(z).$$

Then,

$$c(t) = \mathrm{Res}_{z=0}\,\frac{t\det(1+tA)}{z\det(1-zA)}h(t,z^{-1})dz.$$

Special Cases

Case c_1 :

$$c_1 = \mathrm{Res}_{z=0}\, g_1(z^{-1}) \det(1 - zA)^{-1} z^{-1} dz,$$

where $g_1(z) = \frac{1}{z^{n-1}\partial^{n-1}} f(z)$.

Case c_2 :

$$c_2 = \mathrm{Res}_{z=0} \det(1 - zA)^{-1} \left(g_1(z^{-1})\,\mathrm{tr}\,A - g_2(z^{-1})\right) z^{-1} dz,$$

where $g_2(z) = \frac{1}{z^{n-1}\partial^{n-1}} \left(zf(z) + \frac{n-1}{2\tanh \partial/2} f(z)\right)$.

Infinitesimal Cherednik algebras of $\mathfrak{sp}_{2n}$

Definition

Let V be the standard $2n$-dimensional representation of $\mathfrak{sp}_{2n}$ with symplectic form ω, and let $\alpha : V \times V \to \mathfrak{U}\mathfrak{sp}_{2n}$.
The **infinitesimal Cherednik algebra** H_α is defined as the quotient of $\mathfrak{U}\mathfrak{sp}_{2n} \ltimes T(V)$ by the relations:

$$[x, y] = \alpha(x, y)$$

for all $x, y \in V$. In addition, H_α must satisfy the PBW property.

PBW pairings

- Etingof, Gan, and Ginzburg proved that α is given by $\sum_{j=0}^{k} \alpha_{2j} r_{2j}$ where r_j is the coefficient of z^j in the expansion of

$$\omega(x, (1 - z^2 A^2)^{-1} y) \det(1 - zA)^{-1} = r_0(x, y) + r_2(x, y) z^2 + \cdots .$$

Note that since $A \in \mathfrak{sp}_{2n}$, the expansion $\det(1 - zA)^{-1}$ only contains even powers of z.

Center for $H_\alpha(\mathfrak{sp}_{2n})$

- Let
$$t_i = \sum_{j=1}^{n} [\beta_i, v_j] v_j^*,$$
where β_i is defined by $\sum_{i=1}^{n} \beta_i z^{2i} = \det(1 - zA)$. By v_j^*, we mean the element of V that satisfies $\omega(v_i, v_j^*) = \delta_{ij}$.
- These t_i generate the center of $H_0(\mathfrak{sp}_{2n})$.
- We conjecture that $\mathfrak{z}(H_\alpha) = k[t_1 + c_1, \ldots, t_n + c_n]$, where $c_i \in \mathfrak{z}(\mathfrak{U}\mathfrak{sp}_{2n})$ are unique up to a constant.

Poisson Center for $H_\alpha(\mathfrak{sp}_{2n})$

Theorem

Let $c(t)$ be the generating function for the c_i:

$$c(t) = \sum_{i=1}^{n} (-1)^{i+1} c_i z^{2i}.$$

Then,

$$c(t) = 2\,\mathrm{Res}_{z=0}\, \alpha(z^{-1}) \frac{\det(1-tA)}{\det(1-zA)} \frac{z^{-3}}{1-z^2 t^{-2}} dz.$$

Summary

- Last year, we studied $H_\alpha(\mathfrak{gl}_n)$. We conjectured the formula for the first central element's action on the Verma module.
- Using this conjecture, we calculated the Shapovalov form and determined the finite dimensional irreducible representations.
- We proved this conjecture earlier this year, thereby proving all aforementioned results.
- Currently, we are trying to find the other central elements. To do this, we pass to the Poisson algebra, which is easier to handle.
- Also, we are trying to extend results from $H_\alpha(\mathfrak{gl}_n)$ to $H_\alpha(\mathfrak{sp}_{2n})$, such as Kostant's theorem and the classification of finite dimensional representations.

Acknowledgments

- My mentor, Sasha Tsymbaliuk, for guiding me through this project
- Professor Etingof for suggesting this project
- The PRIMES program for providing this research opportunity