Enumeration of Graded Poset Structures on Graphs

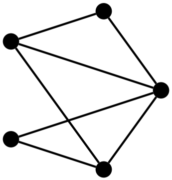
Aaron J Klein

Brookline High School

Second Annual MIT PRIMES Conference
May 19, 2012

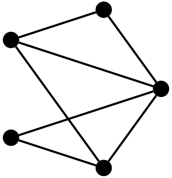
Definitions

A **graph** is a collection of vertices and the edges connecting them.

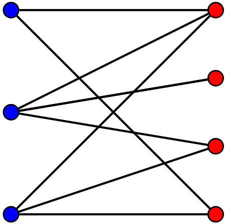


Definitions

A **graph** is a collection of vertices and the edges connecting them.



A **bipartite graph** is a graph whose vertices can be partitioned into two sets such that no edge connects two vertices from the same set.

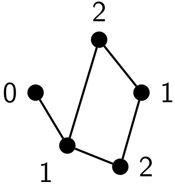


Rankings

A **ranking** of a graph G is an assignment to every vertex $v \in G$ of an integer rank $h(v)$ such that if there is an edge $e \in G$ connecting vertices v_1 and v_2, then $|h(v_1) - h(v_2)| = 1$. Two rankings are considered equivalent if they differ by a constant.

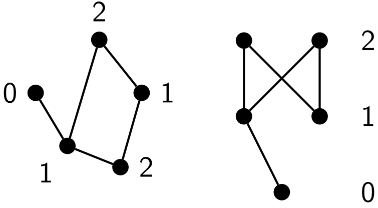
Rankings

A **ranking** of a graph G is an assignment to every vertex $v \in G$ of an integer rank $h(v)$ such that if there is an edge $e \in G$ connecting vertices v_1 and v_2, then $|h(v_1) - h(v_2)| = 1$. Two rankings are considered equivalent if they differ by a constant.



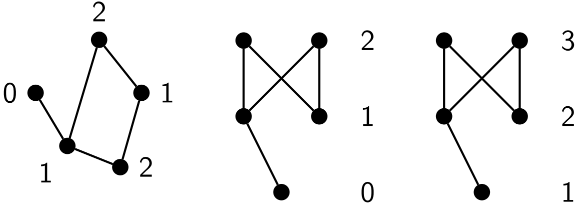
Rankings

A **ranking** of a graph G is an assignment to every vertex $v \in G$ of an integer rank $h(v)$ such that if there is an edge $e \in G$ connecting vertices v_1 and v_2, then $|h(v_1) - h(v_2)| = 1$. Two rankings are considered equivalent if they differ by a constant.



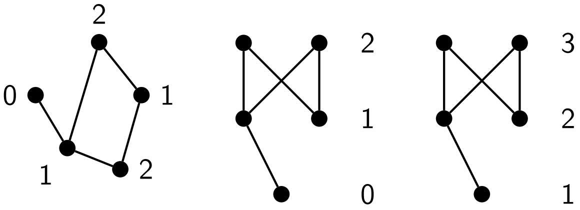
Rankings

A **ranking** of a graph G is an assignment to every vertex $v \in G$ of an integer rank $h(v)$ such that if there is an edge $e \in G$ connecting vertices v_1 and v_2, then $|h(v_1) - h(v_2)| = 1$. Two rankings are considered equivalent if they differ by a constant.



Rankings

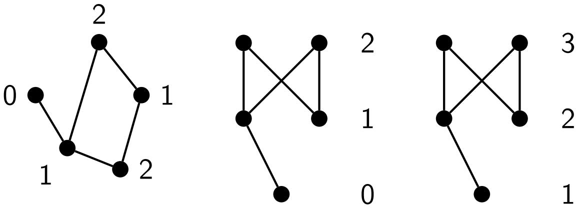
A **ranking** of a graph G is an assignment to every vertex $v \in G$ of an integer rank $h(v)$ such that if there is an edge $e \in G$ connecting vertices v_1 and v_2, then $|h(v_1) - h(v_2)| = 1$. Two rankings are considered equivalent if they differ by a constant.



✎ We denote the number of distinct rankings of a graph G by $\mathcal{R}(G)$.

Rankings

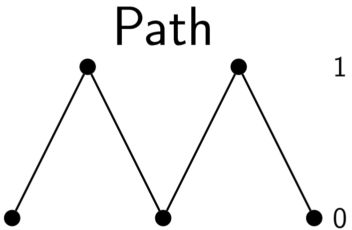
A **ranking** of a graph G is an assignment to every vertex $v \in G$ of an integer rank $h(v)$ such that if there is an edge $e \in G$ connecting vertices v_1 and v_2, then $|h(v_1) - h(v_2)| = 1$. Two rankings are considered equivalent if they differ by a constant.



✎ We denote the number of distinct rankings of a graph G by $\mathcal{R}(G)$.

A graph has at least one ranking ($\mathcal{R}(G) > 0$) if and only if it is a bipartite graph.

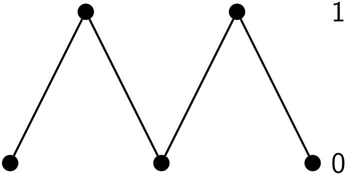
Examples



$$\mathcal{R}(P_n) = 2^{n-1}$$

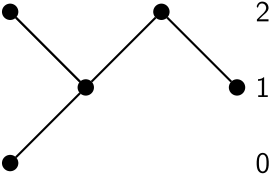
Examples

Path



$\mathcal{R}(P_n) = 2^{n-1}$

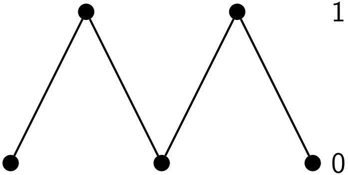
Tree



$\mathcal{R}(\mathcal{T}_n) = 2^{n-1}$

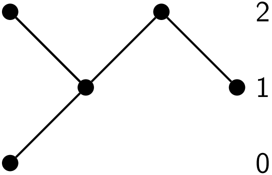
Examples

Path



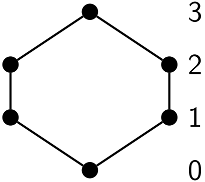
$\mathcal{R}(P_n) = 2^{n-1}$

Tree



$\mathcal{R}(\mathcal{T}_n) = 2^{n-1}$

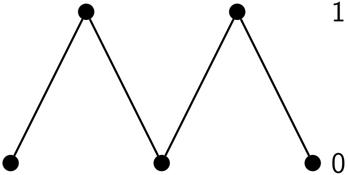
Cycle



$\mathcal{R}(\mathcal{C}_n) = \binom{n}{n/2}$

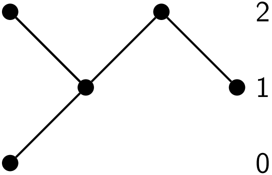
Examples

Path



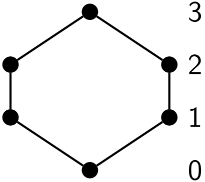
$\mathcal{R}(P_n) = 2^{n-1}$

Tree



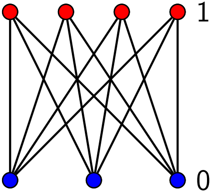
$\mathcal{R}(\mathcal{T}_n) = 2^{n-1}$

Cycle



$\mathcal{R}(\mathcal{C}_n) = \binom{n}{n/2}$

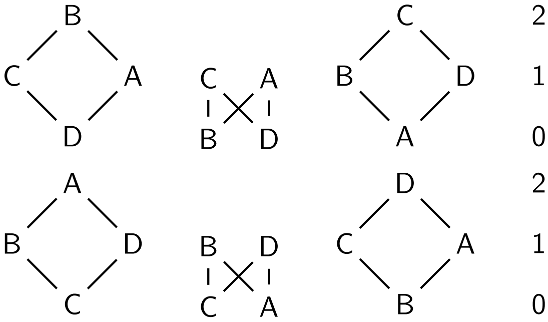
Complete Bipartite



$\mathcal{R}(K_{m,n}) = 2^m + 2^n - 2$

Examples, contd.

4-Cycle



Generating Functions

Theorem

For every graph G, there is a generating function

$$\mathfrak{R}(G) = \prod_{e \in G} \left(\prod_{c \in CYC(G)} {y_c}^{d_e(c)} + \prod_{c \in CYC(G)} {y_c}^{-d_e(c)} \right)$$

whose constant term is equal to $\mathcal{R}(G)$.

Generating Functions

Theorem

For every graph G, there is a generating function

$$\mathfrak{R}(G) = \prod_{e \in G} \left(\prod_{c \in CYC(G)} {y_c}^{d_e(c)} + \prod_{c \in CYC(G)} {y_c}^{-d_e(c)} \right)$$

whose constant term is equal to $\mathcal{R}(G)$.

Example

For a 4-cycle, we have

$$\mathfrak{R}(\mathcal{C}_4) = \prod_{e \in \mathcal{C}_4} \left(y + y^{-1}\right) = \left(y + y^{-1}\right)^4,$$

so $\mathcal{R}(\mathcal{C}_4) = 6$, as we saw in the previous slide.

Generating Functions

Theorem

For every graph G, there is a generating function

$$\mathfrak{R}(G) = \prod_{e \in G} \left(\prod_{c \in CYC(G)} {y_c}^{d_e(c)} + \prod_{c \in CYC(G)} {y_c}^{-d_e(c)} \right)$$

whose constant term is equal to $\mathcal{R}(G)$.

Example

For a 4-cycle, we have

$$\mathfrak{R}(\mathcal{C}_4) = \prod_{e \in \mathcal{C}_4} \left(y + y^{-1}\right) = \left(y + y^{-1}\right)^4,$$

so $\mathcal{R}(\mathcal{C}_4) = 6$, as we saw in the previous slide.

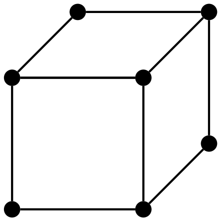
☞ The generating function is not easy to evaluate for general G.

Squarely Generated Graphs

A **squarely generated graph** is a graph whose cycle space is a vector space that can be generated by only 4-cycles.

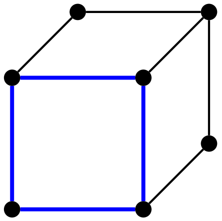
Squarely Generated Graphs

A **squarely generated graph** is a graph whose cycle space is a vector space that can be generated by only 4-cycles.



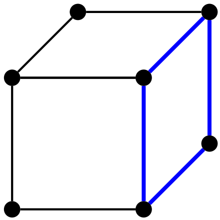
Squarely Generated Graphs

A **squarely generated graph** is a graph whose cycle space is a vector space that can be generated by only 4-cycles.



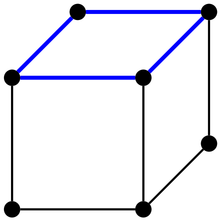
Squarely Generated Graphs

A **squarely generated graph** is a graph whose cycle space is a vector space that can be generated by only 4-cycles.



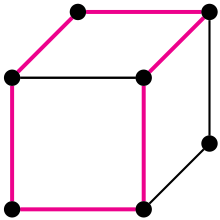
Squarely Generated Graphs

A **squarely generated graph** is a graph whose cycle space is a vector space that can be generated by only 4-cycles.



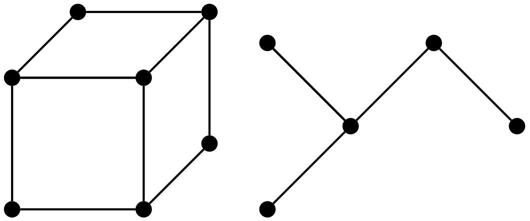
Squarely Generated Graphs

A **squarely generated graph** is a graph whose cycle space is a vector space that can be generated by only 4-cycles.



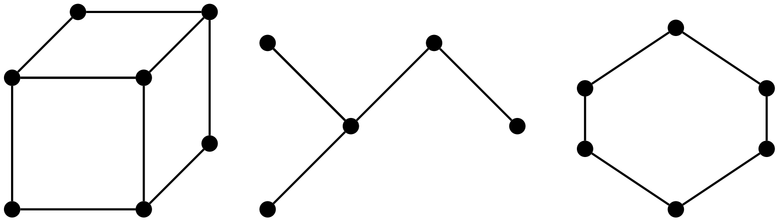
Squarely Generated Graphs

A **squarely generated graph** is a graph whose cycle space is a vector space that can be generated by only 4-cycles.



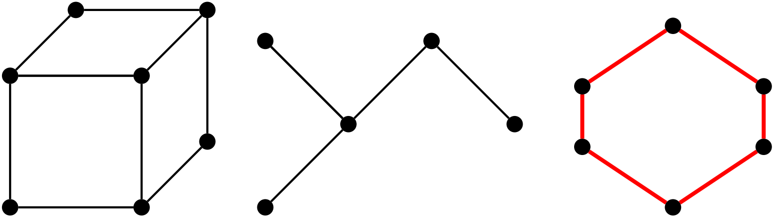
Squarely Generated Graphs

A **squarely generated graph** is a graph whose cycle space is a vector space that can be generated by only 4-cycles.



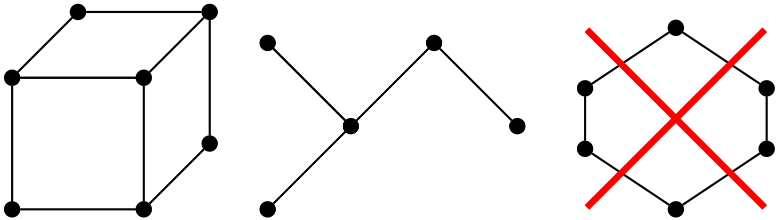
Squarely Generated Graphs

A **squarely generated graph** is a graph whose cycle space is a vector space that can be generated by only 4-cycles.



Squarely Generated Graphs

A **squarely generated graph** is a graph whose cycle space is a vector space that can be generated by only 4-cycles.

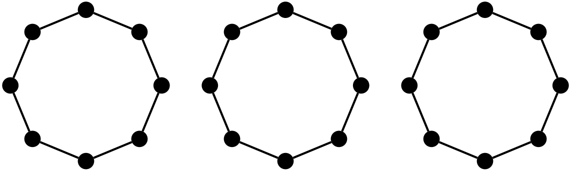


Colorings

For $k \geq 1$, a **proper k-coloring** of a graph G is an assignment to every vertex $v \in G$ of a color $1 \leq c(v) \leq k$ such that no two vertices with the same color are connected by an edge.

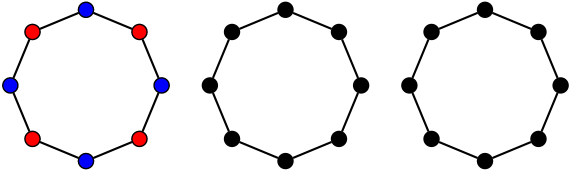
Colorings

For $k \geq 1$, a **proper** k**-coloring** of a graph G is an assignment to every vertex $v \in G$ of a color $1 \leq c(v) \leq k$ such that no two vertices with the same color are connected by an edge.



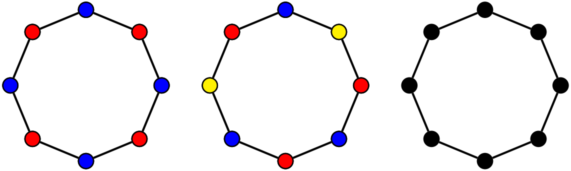
Colorings

For $k \geq 1$, a **proper k-coloring** of a graph G is an assignment to every vertex $v \in G$ of a color $1 \leq c(v) \leq k$ such that no two vertices with the same color are connected by an edge.



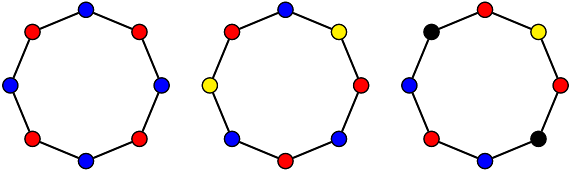
Colorings

For $k \geq 1$, a **proper k-coloring** of a graph G is an assignment to every vertex $v \in G$ of a color $1 \leq c(v) \leq k$ such that no two vertices with the same color are connected by an edge.



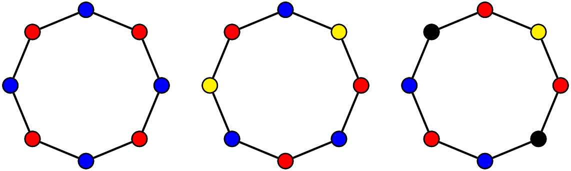
Colorings

For $k \geq 1$, a **proper k-coloring** of a graph G is an assignment to every vertex $v \in G$ of a color $1 \leq c(v) \leq k$ such that no two vertices with the same color are connected by an edge.



Colorings

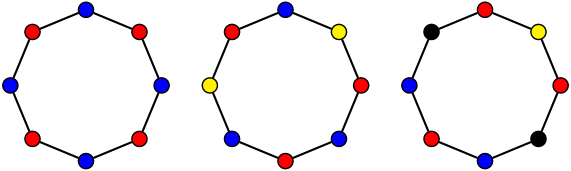
For $k \geq 1$, a **proper k-coloring** of a graph G is an assignment to every vertex $v \in G$ of a color $1 \leq c(v) \leq k$ such that no two vertices with the same color are connected by an edge.



For any graph G, the **chromatic polynomial** $\chi_G(x)$ is a polynomial such that for any given k, $\chi_G(k)$ is the number of proper k-colorings of G.

Colorings

For $k \geq 1$, a **proper k-coloring** of a graph G is an assignment to every vertex $v \in G$ of a color $1 \leq c(v) \leq k$ such that no two vertices with the same color are connected by an edge.



For any graph G, the **chromatic polynomial** $\chi_G(x)$ is a polynomial such that for any given k, $\chi_G(k)$ is the number of proper k-colorings of G.

Example

For the cycle $\mathcal{C}_{2n}$, the chromatic polynomial is $\chi_{\mathcal{C}_{2n}}(x) = (x-1)^{2n} + x - 1$

Rank-Color Duality

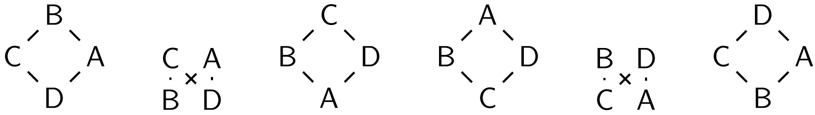
Theorem

If G is a squarely generated graph, then there is a direct correspondence between its rankings and colorings such that $\mathcal{R}(G) = \frac{1}{3}\chi_G(3)$.

Rank-Color Duality

Theorem

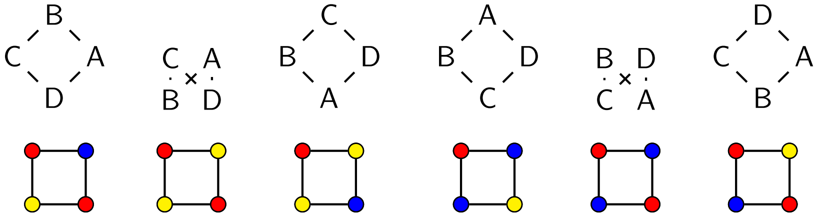
If G is a squarely generated graph, then there is a direct correspondence between its rankings and colorings such that $\mathcal{R}(G) = \frac{1}{3}\chi_G(3)$.



Rank-Color Duality

Theorem

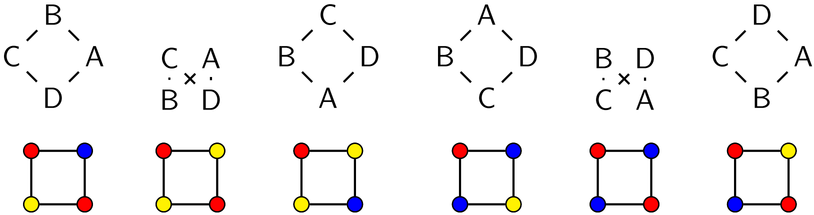
If G is a squarely generated graph, then there is a direct correspondence between its rankings and colorings such that $\mathcal{R}(G) = \frac{1}{3}\chi_G(3)$.



Rank-Color Duality

Theorem

If G is a squarely generated graph, then there is a direct correspondence between its rankings and colorings such that $\mathcal{R}(G) = \frac{1}{3}\chi_G(3)$.



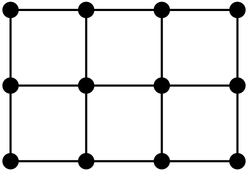
This is useful because chromatic polynomials are much more well-studied than rankings.

Grid Graphs

A **grid graph** is a graph $\mathcal{L}_{m,n}$ whose vertices are all (i, j) for $1 \leq i \leq m$ and $1 \leq j \leq n$ with edges connecting (i, j) to $(i, j + 1)$ and $(i + 1, j)$.

Grid Graphs

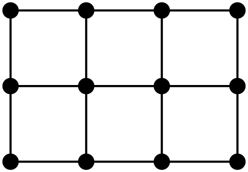
A **grid graph** is a graph $\mathcal{L}_{m,n}$ whose vertices are all (i, j) for $1 \leq i \leq m$ and $1 \leq j \leq n$ with edges connecting (i, j) to $(i, j + 1)$ and $(i + 1, j)$.



$\mathcal{L}_{3,4}$

Grid Graphs

A **grid graph** is a graph $\mathcal{L}_{m,n}$ whose vertices are all (i, j) for $1 \leq i \leq m$ and $1 \leq j \leq n$ with edges connecting (i, j) to $(i, j + 1)$ and $(i + 1, j)$.

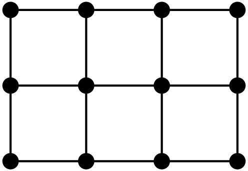


$\mathcal{L}_{3,4}$

- $\mathcal{R}(\mathcal{L}_{2,n}) = 2 \cdot 3^{n-1}$

Grid Graphs

A **grid graph** is a graph $\mathcal{L}_{m,n}$ whose vertices are all (i, j) for $1 \leq i \leq m$ and $1 \leq j \leq n$ with edges connecting (i, j) to $(i, j+1)$ and $(i+1, j)$.

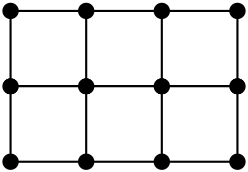


$\mathcal{L}_{3,4}$

- $\mathcal{R}(\mathcal{L}_{2,n}) = 2 \cdot 3^{n-1}$
- $\mathcal{R}(\mathcal{L}_{3,n}) = \frac{17+3\sqrt{17}}{34}\left(\frac{5+\sqrt{17}}{2}\right)^n + \frac{17-3\sqrt{17}}{34}\left(\frac{5-\sqrt{17}}{2}\right)^n$

Grid Graphs

A **grid graph** is a graph $\mathcal{L}_{m,n}$ whose vertices are all (i,j) for $1 \leq i \leq m$ and $1 \leq j \leq n$ with edges connecting (i,j) to $(i,j+1)$ and $(i+1,j)$.



$\mathcal{L}_{3,4}$

- $\mathcal{R}(\mathcal{L}_{2,n}) = 2 \cdot 3^{n-1}$
- $\mathcal{R}(\mathcal{L}_{3,n}) = \frac{17+3\sqrt{17}}{34}\left(\frac{5+\sqrt{17}}{2}\right)^n + \frac{17-3\sqrt{17}}{34}\left(\frac{5-\sqrt{17}}{2}\right)^n$
- For general m and n, there is no known closed-form formula for $\mathcal{R}(\mathcal{L}_{m,n})$. However, for any particular m and n, $\mathcal{R}(\mathcal{L}_{m,n})$ can be calculated using the transform matrix method.

Future Work

- Use physics ideas (e.g. Potts Model) as help in finding formulae for squarely generated and especially grid graphs.

Future Work

- Use physics ideas (e.g. Potts Model) as help in finding formulae for squarely generated and especially grid graphs.
- Try to further understand the generating function for general G.

Future Work

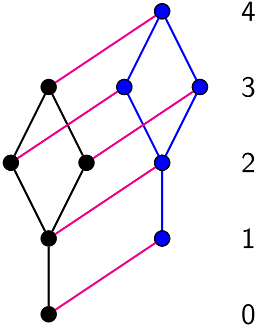
- Use physics ideas (e.g. Potts Model) as help in finding formulae for squarely generated and especially grid graphs.
- Try to further understand the generating function for general G.
- Find families of graphs for which the generating function is easily evaluable, such as $G \times \mathcal{E}$, where $\mathcal{E}$ is a single edge.

Future Work

- Use physics ideas (e.g. Potts Model) as help in finding formulae for squarely generated and especially grid graphs.
- Try to further understand the generating function for general G.
- Find families of graphs for which the generating function is easily evaluable, such as $G \times \mathcal{E}$, where $\mathcal{E}$ is a single edge.

Future Work

- Use physics ideas (e.g. Potts Model) as help in finding formulae for squarely generated and especially grid graphs.
- Try to further understand the generating function for general G.
- Find families of graphs for which the generating function is easily evaluable, such as $G \times \mathcal{E}$, where $\mathcal{E}$ is a single edge.



I would like to thank

I would like to thank

- The MIT PRIMES program

I would like to thank

- The MIT PRIMES program
- My mentor Yan Zhang

I would like to thank

- The MIT PRIMES program
- My mentor Yan Zhang
- Professor Richard Stanley

I would like to thank

- The MIT PRIMES program
- My mentor Yan Zhang
- Professor Richard Stanley
- Tanya Khovanova, Alan Zhou, and Steven Sam