On the Winning and Losing Parameters of Schmidt's Game

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Abstract

First introduced by Wolfgang Schmidt, the (α, β)-game and its modifications have been shown to be a powerful tool in Diophantine approximation, metric number theory, and dynamical systems. However, natural questions about the winning-losing parameters of most sets have not been studied thoroughly even after more than 40 years. There are a few results in the literature showing that some non-trivial points and small regions are winning or losing, but complete pictures remain largely unknown. Our main goal in this paper is to provide as much detail as possible about the global pictures of winning-losing parameters for some interesting families of sets.

1 Introduction

In [S1] (1966), Wolfgang Schmidt proved that the set of badly approximable numbers has full Hausdorff dimension and the countable intersection property, using the (α, β)-game as the main tool. The result was then generalized to linear forms and matrices in [S2], using essentially the same method. The (α, β)-game has been shown to be powerful method to prove the existence and abundance of sets arising in dynamical systems, Diophantine approximations and metric number theory (see, for instance, [Da, KW, Mc]). In this section, we will describe Schmidt's (α, β)-game, present some notations used throughout the paper, and summarize our results.

Let $I = \{(\alpha, \beta) : 0 < \alpha, \beta < 1\} = (0, 1) \times (0, 1)$ denote the open unit square, and $\mathbf{B}(c, r) = [c - r, c + r] = \{x \in \mathbb{R} : |x - c| \leq r\}$ be the closed interval (ball) of radius r centered at c. For each pair $(\alpha, \beta) \in I$, consider the following *Schmidt (α, β)-game* on $\mathbb{R}$, played by two players, called Alice and Bob. The game starts with Bob choosing a closed interval $B_0 = \mathbf{B}(b_0, \frac{d}{2})$ for some $d > 0$. The game then proceeds inductively with Alice and Bob alternatively take turns, choosing closed intervals A_n's, B_n's respectively, to form a nested sequence:

$$B_0 \supseteq A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \cdots ,$$

such that the length $|A_n|$ of A_n is equal to $\alpha|B_{n-1}|$ and the length $|B_n| = \beta|A_n|$. In other words,

$$A_n = \mathbf{B}\left(a_n, \frac{d\alpha(\alpha\beta)^{n-1}}{2}\right) \text{ with } |a_n - b_{n-1}| \le (1-\alpha)\frac{d(\alpha\beta)^{n-1}}{2},$$
$$B_n = \mathbf{B}\left(b_n, \frac{d(\alpha\beta)^{n}}{2}\right) \text{ with } |b_n - a_n| \le (1-\beta)\frac{d\alpha(\alpha\beta)^{n-1}}{2}.$$

A subset S of $\mathbb{R}$ is called *(α,β)-winning* if Alice can play in such a way that the unique point of intersection:

$$x_\infty := \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} A_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

lies in S, no matter how Bob plays. In other words, for any $n \in \mathbb{N}$ and for any choice of B_n made by Bob, Alice can choose A_n in such a way that $x_\infty \in S$. We will say that S is *(α,β)-losing* if it is not (α,β)-winning. In [S1], it was shown that a winning or losing strategy for an (α,β)-game need only be dependent on the last interval which the opponent played, as opposed to all of the opponents' previous intervals. Let us define $\mathscr{B}$ as the set of all intervals in $\mathbb{R}$ of positive length. Then, we may consider a winning or losing strategy $f : \mathscr{B} \to \mathscr{B}$ as a function which describes what interval Alice or Bob, respectively, would choose given their opponents previous interval, for all intervals in $\mathbb{R}$, so that they could win.

For $S \subset \mathbb{R}$, define the *Schmidt diagram* $D(S)$ of S to be the set of pairs $(\alpha,\beta) \in I$ such that S is (α,β)-winning. Let

$$\hat{D} := \left\{(\alpha,\beta) \in I : \beta > 2 - \frac{1}{\alpha}\right\}$$

and

$$\check{D} := \left\{(\alpha,\beta) \in I : \beta \ge \frac{1}{2-\alpha}\right\}.$$

Then $\hat{D}$ and $\check{D}$ are Schmidt diagrams of any co-countable set and any dense countable set respectively (see [S1]). Besides $I, \emptyset, \hat{D}$, and $\check{D}$ (proposition 2.1), there are no other known examples of sets of the form $D(S)$ for some S. In this paper, we show large portions of other Schmidt diagrams that are not in the preceding four forms.

In the next section, we will show some basic properties of Schmidt diagrams, and a technical lemma which will be used in later sections. These properties include details on portions of the Schmidt diagram that are trivial for certain sets and the technical lemma is used to show Bob or

Alice's ability to force his/her opponent to move into a particular interval. In section 3, we describe in great detail the Schmidt diagram of the following family of sets:

$$F_{b,w} := \left\{ x \in \mathbb{R} : \begin{array}{c} \text{there are at most finitely many } w\text{'s} \\ \text{in the base } b \text{ expansion of } x \end{array} \right\},$$

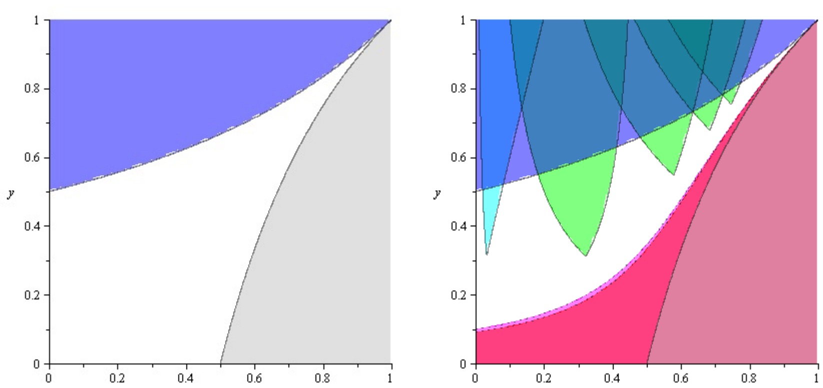
where $b = 2, 3, 4, ...$ be a base expansion, and $w \in \{0, 1, ..., b-1\}$ is any digit of base b. It is known that these sets all have measure zero and are dense. We have two separate portions, one devoted to winning regions and one devoted to losing regions. Our basic strategy is to consider each digit individually to see whether Alice can avoid making that digit w. For our losing arguments, we employ similar tactics to see how Bob can force various digits to be w. A significant difference will arise between the two strategies, though because Bob needs to only ensure that infinitely many digits are w, while Alice needs to ensure that from a certain point on, all digits are not w. The winning and losing theorems for $F_{b,w}$ which we found were:

Theorem 3.1. Let $W_m = \left\{ (\alpha, \beta) \in I : \frac{1}{b} \leq (\alpha\beta)^m \leq \frac{\alpha}{2(1-\alpha)} \frac{\beta(b-1)(1-2\alpha+\alpha\beta)-(1-2\beta+\alpha\beta)}{1+\alpha\beta(b-1)} \right\}$, for $m \in \mathbb{N}$. Then $\bigcup_{m=1}^{\infty} W_m \subset D(F_{b,w})$.

Theorem 3.2. If $\alpha > \frac{b\beta+\beta-1}{2b\beta-b\beta^2-\beta+\beta^2}$, then $F_{b,w}$ is (α, β)-losing.

Theorem 3.3. If $\alpha > \frac{(b-1)\beta+\beta-1}{2(b-1)\beta-(b-1)\beta^2-\beta+\beta^2}$, then $F_{b,0}$ is (α, β)-losing.

Here are graphs of the trivial zones (as described in the following section) and of what we found of the Schmidt Diagram for $F_{b,w}$:



The graph on the left is the trivial zones, and the graph on the right includes our winning and losing zones and is for $F_{10,w}$. On the right graph, the greens and blues are winning, while the purples and reds are losing. The blue tooth-like shape is a distortion of the big green tooth to its right, due to the fact that it was proved in [S1] that if (α, β) is winning, $(\alpha(\alpha\beta)^n, \beta)$ is also winning. There is a winning zone for $F_{b,w}$ given in [S1], but it does not show up in the Schmidt diagram for $b = 10$ (the winning zones tend to expand as b increases, while the losing zones retract as b increases).

Finally, in section 4, we will apply the results and methods in sections 2 and 3 to provide a big picture of winning and losing parameters of a family of sets arising from [Dr, Fr]. Namely, the set

$$C_b := \left\{ x \in \mathbb{R} : \begin{array}{c} \text{there is at least one } 0 \text{ or } b-1 \\ \text{in the base b expansion of } x \end{array} \right\}.$$

Our losing result for C_b is an extension of Freiling's original argument in [Fr], which only included two points, and our winning argument uses the ideas from our work with $F_{b,w}$. In [Dr, Fr], it was shown that C_b exhibits some very interesting properties, which our winning results will shed some light on. The key fact behind these properties is in fact that the denominator of the quantity $\log_b \alpha\beta$ (if it is even rational at all) plays a large role in which player has the advantage. The winning and losing theorems for C_b which we found were:

Theorem 4.1. (1) C_6 is $(\frac{1}{2}, \frac{1}{2})$-winning but $(\frac{1}{2}, \frac{1}{3})$-losing.

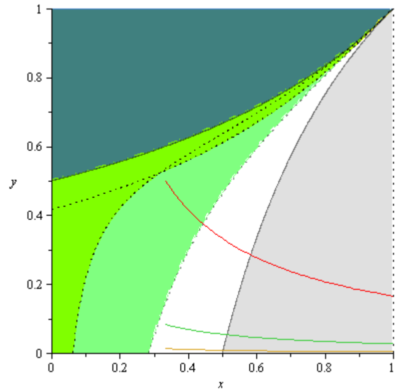
(2) C_4 is $(\frac{1}{2}, \frac{1}{2})$-losing but (α, β)-winning for $\max\left(\left|\frac{1}{2} - \alpha\right|, \left|\frac{1}{2} - \beta\right|\right) < \epsilon, 0 < |\ln(4\alpha\beta)| \leq \epsilon$, where $\epsilon = 2^{-12}$.

Theorem 4.2. If $\alpha\beta = \frac{1}{b}$ and $\alpha \geq \frac{2}{b}$, then C_b is (α, β)-losing. Additionally, it is $(\alpha, \beta(\frac{1}{b})^n)$-losing for all $n \in \mathbb{N}$.

Theorem 4.3. If $\beta > \frac{(b+1)\alpha - 2}{\alpha((2b-4)-(b-3)\alpha)}$ and $\log_b \alpha\beta \notin \mathbb{Q}$, then C_b is (α, β)-winning.

Theorem 4.4. If $\log_b \alpha\beta = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{Z}, p, q$ relatively prime, and $\dfrac{1-\alpha}{\frac{(b-1)\alpha}{2} - \frac{(b-1)\alpha\beta(1-\alpha)}{1-\alpha\beta}} > \frac{1}{\alpha\beta}$ or $\dfrac{1-\alpha}{\frac{(b-1)\alpha}{2} - \frac{(b-1)\alpha\beta(1-\alpha)}{1-\alpha\beta}} > b^{\frac{1}{q}}$, then C_b is (α, β)-winning.

Here is a graph of what we found of the Schmidt Diagram of C_b (the graph shown uses $b = 6$) which shows points which are losing (the colored lines), a region in which all points are winning (the yellow region), and a region in which points such that $\log_b \alpha\beta \notin \mathbb{Q}$ are winning (the green region).



The (α, β) game is interesting to study because it is an infinite game and produces some counterintuitive results. For instance, Chris Freiling in [Fr] answered a question posed in [S1] and showed that decreasing α and keeping β constant can allow Bob to win. In this paper, we made significant progress in determining another complete Schmidt Diagram, and came up with innovative methods for proving that large regions of the Schmidt diagram are (α, β) winning and losing.

Remark: Our definition of losing is slightly different from what is usually defined in infinite game theory, where the definition of losing is that Bob has a winning strategy. Since the sets that we study are Borel sets, by Martin Theorem [Ma], both concepts agree in our context.

2 Miscellaneous Results

The following cases, which were also discussed in [S1], are said to have trivial Schmidt diagrams:

Proposition 2.1. *(1) If S is not dense, then $D(S) = \emptyset$.*

(2) $D(\mathbb{R}) = I$.

(3) If S is dense, then $\check{D} \subseteq D(S)$. Equality happens if S is also countable.

(4) If $S \neq \mathbb{R}$, then $D(S) \subseteq \hat{D}$. Equality happens if S is also co-countable.

Proof. (1) If S is not dense, clearly Bob can choose B_1 such that $S \cap B_1 = \emptyset$ for any (α, β), in which case he immediately wins.

(2) Trivial.

(3) Let S be a dense set, then $S \cap \mathbf{B}(b_0, (1-\alpha)\frac{d}{2}) \neq \emptyset$. Pick $a \in \mathbf{B}(b_0, (1-\alpha)\frac{d}{2})$, then Alice can choose her center a_1 of A_1 to be a, and $A_1 = \mathbf{B}(a, \frac{d\alpha}{2}) \subseteq B_0$. If $\beta \geq \frac{1}{2-\alpha}$, then $2\beta - 1 \geq \alpha\beta > 0$. Therefore, even if Bob moves as far to one side on a given turn as possible, his interval will be $B_n = \mathbf{B}(a_n \pm (1-\beta)\frac{d\alpha(\alpha\beta)^{n-1}}{2}, \frac{d(\alpha\beta)^n}{2}) \supseteq \mathbf{B}(a_n, (2\beta-1)\frac{d\alpha(\alpha\beta)^{n-1}}{2}) \supseteq \mathbf{B}(a_n, \frac{d\alpha(\alpha\beta)^n}{2})$.

In particular, Alice may fix her centers $a_{n+1} = a_n = ... = a_1 = a$. That implies no matter how Bob chooses $B_1, B_2, ...,$

$$x_\infty = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a = a \in S.$$

Now if S is a countable dense set, all points outside $\check{D}$ are losing, since Alice cannot center her intervals around any specific point, allowing Bob to systematically eliminate the points in S in a countable way from his intervals.

(4) Similarly, if $S \neq \mathbb{R}$, Bob can fix the centers of his intervals at a point not in S, as long as $1 - 2\alpha + \alpha\beta \leq 0$ or equivalently $\beta \leq 2 - \frac{1}{\alpha}$. And if S^c is countable, Alice can systematically eliminate the points in S^c, making all points in $\hat{D}$ winning. □

Some of the simple results about (α,β)-winning are summarized in the following proposition:

Proposition 2.2. *Let $S, S_1, S_2 \subset \mathbb{R}$.*

(1) If $S_1 \subseteq S_2$ and S_1 is (α,β)-winning, then S_2 is also (α,β)-winning, i.e., $D(S_1) \subseteq D(S_2)$.

(2) $D(S_1 \cap S_2) \subseteq D(S_1) \cap D(S_2)$ and $D(S_1 \cup S_2) \supseteq D(S_1) \cup D(S_2)$.

(3) If S is (α,β)-winning then $S^c = \mathbb{R} \setminus S$ is (β,α)-losing. In other words, $D(S) \subseteq \sigma(I \setminus D(S^c))$ where $\sigma(\alpha,\beta) = (\beta,\alpha)$.

(4) If F is a locally finite set and $S \cup F \neq \mathbb{R}$, then $D(S \cup F) = D(S) = D(S \setminus F)$.

(5) If $S' = kS + c$ for some $k, c \in \mathbb{R}$, $k \neq 0$, then $D(S') = D(S)$.

Proof. (1): If S_2 is (α,β)-losing, then on the (α,β)-game on S_1, Bob may employ the same strategy, and obviously the point of convergence will not be in S_1, since it is not in S_2.

(2): Any point not in S_1 or S_2 will not be in $S_1 \cap S_2$, and thus any winning strategy that Bob has for a given (α,β) for S_1 or S_2 will apply for $S_1 \cap S_2$ as well. Similarly, any winning strategy Alice has for a given (α,β) for S_1 or S_2 will apply for $S_1 \cup S_2$ as well.

(3): If S is (α, β)-winning, then Alice has a winning strategy $f(B)$ playing on S. So playing (β, α)-game on S^c, Bob can win using Alice's strategy $f(B)$. And thus, S^c is (β, α)-losing.

(4): If S is (α, β)-losing, clearly $S \setminus F$ is also (α, β)-losing. If S is (α, β)-winning, then for the (α, β)-game on $S \setminus F$, Alice may systematically exclude the finite points in $F \cap B_0$ from her interval (this is possible since S was not $\mathbb{R}$, but also (α, β)-winning, meaning $1 - 2\alpha + \alpha\beta > 0$ and Bob therefore cannot center his intervals around one point). After she has eliminated all of these points, she can use the same strategy she would in the (α, β)-game on S, as if she were given the interval she has as her first interval on the (α, β)-game on S. Since we have now shown that $D(S) = D(S \setminus F)$, it follows quite easily that $D(S \cup F) = D((S \cup F) \setminus (F \setminus S)) = D(S)$.

(5): If $f(B)$ is a winning strategy for Alice for an (α, β)-game on S, then we can define a winning strategy $g(B)$ for her for the (α, β)-game on S' as $g(B) = kf\left(\frac{1}{k}(B - c)\right) + c$. Similarly, Bob has a corresponding winning strategy on S' for each (α, β) for which he has a winning strategy on S. □

We will end this section by proving a technical lemma on the enforcing power of each player. This Lemma is crucial in our strongest winning and losing theorems:

Lemma 2.3. *If $B_k = \mathbf{B}(b_k, \rho)$, Bob can ensure that*

$$B_{k+n} \subseteq \mathbf{B}\left(b_k, \rho - 2\rho\alpha(1-\beta)\frac{1-(\alpha\beta)^n}{1-\alpha\beta}\right).$$

Similarly, if $A_k = \mathbf{B}(a_k, \rho)$, Alice can ensure that

$$A_{k+n} \subseteq \mathbf{B}\left(a_k, \rho - 2\rho\beta(1-\alpha)\frac{1-(\alpha\beta)^n}{1-\alpha\beta}\right).$$

Proof. Clearly, it holds for $n = 0$. Then, given $B_{k+n} \subseteq \mathbf{B}\left(b_k, \rho - \frac{2\rho\alpha(1-\beta)(1-(\alpha\beta)^n)}{1-\alpha\beta}\right)$, should Alice move as far in one direction as possible, her endpoint which is closer to b_k, would be bounded by

$$b_k \pm \left(\rho - \frac{2\rho\alpha(1-\beta)(1-(\alpha\beta)^n)}{1-\alpha\beta} - 2\rho\alpha(\alpha\beta)^n\right).$$

And if Bob moves as far as possible back towards b_k, his further endpoint would be bounded by:

$$b_k \pm \left(\rho - \frac{2\rho\alpha(1-\beta)(1-(\alpha\beta)^n)}{1-\alpha\beta} - 2\rho\alpha(\alpha\beta)^n + 2\rho(\alpha\beta)^{n+1}\right) =$$
$$= b_k \pm \left(\rho - \frac{2\rho\alpha(1-\beta)(1-(\alpha\beta)^{n+1})}{1-\alpha\beta}\right).$$

Therefore, the lemma follows by induction. □

3 Schmidt's diagram of $F_{b,w}$

Fix a base $b > 1$ and a b-ary digit w, and for each $k \in Z$, let

$$Z_k := \{x \in \mathbb{R} : \text{ the } k^{\text{th}}\ b\text{-ary digit of } x \text{ is } w\}.$$

3.1 Winning pairs

In this section, we will consider for what (α, β) pairs the set $F_{b,w}$ is winning. First, notice that $x \in F_{b,w}$ if and only if there exists a positive integer k_0 such that

$$x \notin \bigcup_{k \geq k_0} Z_k.$$

So, if there is a strategy for Alice such that for some $k_0 > 0$, Alice may disjoint her intervals from that Z_k for all $k \geq k_0$, then Alice will win. We utilize the geometric structure of the Z_k's, in that each Z_k is composed of half-open intervals of length b^{-k} whose centers are each b^{-k+1} apart, to help us consider how Alice may avoid each of the Z_k intervals systematically. For this proof, we first fix m, the number of turns which Alice is allowed to disjoint herself from each succesive Z_k. For each k we consider two conditions for Alice to avoid the Z_k. First, the region in which Alice can ensure her interval will fall after m turns (we utilize Lemma 2.3 for this) is small enough to fit between two consecutive Z_k intervals. Next, we consider the worst scenario which Alice could be presented with: a Z_k lies directly in the center of her interval before the m turns. Then, the interval she was given before the m turns must be large enough that she has room to dodge this Z_k. Thus for each k we have set upper and lower bounds on the size of the interval with which she can begin attempting to avoid the Z_k. Since the size of her interval depends on n_k, the turn which she begins attempting to avoid Z_k, we must find an n_k for every k that satisfies our two conditions. To ensure the existence of such an n_k, we will set the ratio of our upper and lower bound to be greater than or equal to $\frac{1}{\alpha\beta}$. Thus for a fixed k, varying n_k should yield some integer solution (because n_k is the exponent of $\alpha\beta$). This ratio being greater than $\frac{1}{\alpha\beta}$ will be the first of our two overall conditions on α and β. The second will be that $(\alpha\beta)^m \geq \frac{1}{b}$, so that consecutive n_k are more than m apart. In this manner, we shall prove the following theorem.

Theorem 3.1. *Let* $W_m = \left\{(\alpha,\beta) \in I : \frac{1}{b} \leq (\alpha\beta)^m \leq \frac{\alpha}{2(1-\alpha)} \frac{\beta(b-1)(1-2\alpha+\alpha\beta)-(1-2\beta+\alpha\beta)}{1+\alpha\beta(b-1)}\right\}$, *for* $m \in \mathbb{N}$. *Then* $\bigcup_{m=1}^{\infty} W_m \subset D(F_{b,w})$.

Proof. Let $(\alpha,\beta) \in W_m \setminus \check{D}$. Since

$$(\alpha\beta)^m \leq \frac{\alpha}{2(1-\alpha)} \frac{\beta(b-1)(1-2\alpha+\alpha\beta)-(1-2\beta+\alpha\beta)}{1+\alpha\beta(b-1)} \implies$$

$$\frac{1+\dfrac{2\alpha\beta(1-\alpha)(1-(\alpha\beta)^{m-1})}{1-\alpha\beta}-2\alpha}{\dfrac{\alpha-\frac{2\alpha\beta(1-\alpha)(1-(\alpha\beta)^{m-1})}{1-\alpha\beta}}{b-1}} \geq \frac{1}{\alpha\beta},$$

for some $k_0 \in \mathbb{N}$, every value of $d > 0$, and every value of $k \in \mathbb{N}$, there exists some $n_k \in \mathbb{N}$ such that

$$\frac{\alpha-\frac{2\alpha\beta(1-\alpha)(1-(\alpha\beta)^{m-1})}{1-\alpha\beta}}{b-1} \leq \frac{b^{-k-k_0}}{d(\alpha\beta)^{n_k-1}} \leq 1+\frac{2\alpha\beta(1-\alpha)(1-(\alpha\beta)^{m-1})}{1-\alpha\beta}-2\alpha.$$

Then, for all $k \in \mathbb{N}$, Alice can ensure that some A_{n_k+m-1} is disjoint from Z_{k+k_0} since after choosing A_{n_k}, she may ensure that $A_{n_k+m-1} \subset \mathbf{B}(c, \frac{d\alpha(\alpha\beta)^{n_k-1}}{2} - \frac{d\alpha(\alpha\beta)^{n_k-1}\beta(1-\alpha)(1-(\alpha\beta)^{m-1})}{1-\alpha\beta})$, where c is the center of A_{n_k}, and this interval can always fit between two intervals of Z_{k+k_0}, as

$$d\alpha(\alpha\beta)^{n_k-1} - \frac{2d\alpha(\alpha\beta)^{n_k-1}\beta(1-\alpha)(1-(\alpha\beta)^{m-1})}{1-\alpha\beta} \leq (b-1)b^{-k-k_0}.$$

Also, since the farthest from the center of B_{n_k-1} that c can be is $\frac{d(\alpha\beta)^{n_k-1}}{2} - \frac{d\alpha(\alpha\beta)^{n_k-1}}{2}$, even if an interval of Z_{k+k_0} is directly in the center of the B_{n_k-1}, the closer endpoint of the interval Alice can ensure A_{n_k+m-1} is in is farther than the farther endpoint of the Z_{k+k_0} interval, since

$$\frac{d(\alpha\beta)^{n_k-1}}{2} - d\alpha(\alpha\beta)^{n_k-1} + \frac{d\alpha(\alpha\beta)^{n_k-1}\beta(1-\alpha)(1-(\alpha\beta)^{m-1})}{1-\alpha\beta} \geq \frac{b^{-k-k_0}}{2}.$$

(If in either of the conditions, equality holds and Alice is forced to have the right endpoint of A_{n_k+m-1} coincide with the left endpoint of a Z_k interval, which is the closed endpoint, Alice may simply make sure that her next move does not include this endpoint, so she is fully disjoint for Z_k.) Additionally, if $c_1 = 1 + \frac{2\alpha\beta(1-\alpha)(1-(\alpha\beta)^{m-1})}{1-\alpha\beta} - 2\alpha$, then

$$b^{-k-1-k_0} \leq b^{-k-k_0}(\alpha\beta)^m \leq c_1 d(\alpha\beta)^{n_k-1}(\alpha\beta)^m \implies \frac{b^{-k-1-k_0}}{d(\alpha\beta)^{n_k+m-1}} \leq c_1$$

Since the expression $\frac{b^{-k-1-k_0}}{d(\alpha\beta)^{n_{k+1}-1}}$ increases as n_{k+1} increases, there exists some n_{k+1} which satisfies the inequality mentioned previously, and $n_k + m$ is not too large to satisfy the inequality, there must be an $n_{k+1} \geq n_k + m$ that satisfies it. Thus, Alice can repeat this process for all Z_{k+k_0}, and thus the point of convergence will be in $F_{b,w}$. □

3.2 Losing pairs

We now similarly consider losing pairs. Our strategy for Bob to win (and therefore Alice to lose) is to show that he can ensure that the point of convergence (and thus one of his intervals) is contained in infinitely many Z_k's. Again, in order for Bob to contain one of his intervals in a Z_k, he must satisfy two conditions: he must be able to center around a Z_k interval and then use Lemma 2.3 to move inside the interval. Unlike with the winning pairs however, Bob does not need an n for every k to satisfy the bounds, but rather just an infinite number of (n, k) pairs. Therefore, we only need the upper bound to be greater than the lower bound. Then, if $\log_b \alpha\beta$ is irrational, all values in between those bounds will be achieved exactly once (giving an infinite number of (n, k) pairs). If $\log_b \alpha\beta$ is rational, on the other hand, certain values will be achieved infinitely many times, but since Bob has the choice of d he may ensure that these values lie between the bounds (thus giving an infinite number of (n, k) pairs). By this method, we show that Bob can contain the point of convergence in an infinite number of Z_k's and thus win.

Theorem 3.2. *If $\alpha > \frac{b\beta+\beta-1}{2b\beta-b\beta^2-\beta+\beta^2}$, then $F_{b,w}$ is (α, β)-losing.*

Proof. Let $(\alpha, \beta) \in I$ such that $\alpha > \frac{b\beta+\beta-1}{2b\beta-b\beta^2-\beta+\beta^2}$. Then, since $1 - \frac{2\alpha(1-\beta)}{1-\alpha\beta} < \frac{1-\beta}{b\beta}$ (this comes from our initial assumption when not solved for α), there are infinitely many ordered pairs (n, k) such that

$$1 - \frac{2\alpha(1-\beta)}{1-\alpha\beta} < \frac{b^{-k}}{d(\alpha\beta)^n} < \frac{1-\beta}{b\beta}.$$

(If $\log_b \alpha\beta$ is rational, proper choice of d is necessary.) Then, for each pair, Bob can ensure that $B_{n+m_n} \subset Z_k$ for some m_n. This is because he can center B_n around the center of a Z_k interval, since the interval of possible centers of B_n has length $d\alpha(\alpha\beta)^{n-1}(1-\beta) > b^{-k+1}$, it must contain the center of some Z_k interval which are all spaced b^{-k+1} apart. Also, he can ensure that $B_{n+m_n} \subset \mathbf{B}(c, d(\alpha\beta)^n - \frac{2d(\alpha\beta)^n\alpha(1-\beta)(1-(\alpha\beta)^{m_n})}{1-\alpha\beta})$, where c is the common center of B_n and the Z_k interval. Since this interval shares a center with the Z_k interval and is smaller than it, as

$$\lim_{m_n\to\infty} d(\alpha\beta)^n - \frac{2d(\alpha\beta)^n\alpha(1-\beta)(1-(\alpha\beta)^{m_n})}{1-\alpha\beta} = d(\alpha\beta)^n - \frac{2d(\alpha\beta)^n\alpha(1-\beta)}{1-\alpha\beta} < b^{-k},$$

it is a subset of Z_k, meaning B_n must be as well. Therefore, Bob can ensure that the point of convergence is in infinitely many Z_k's, so it is not in $F_{b,w}$. □

Now, we consider the case of $w = 0$ and apply a similar argument to the preceding one, but with Bob trying to contain his intervals in $G_k = \bigcup_{j=k}^{\infty} Z_j$. It can be noted that for $F_{b,0}$ since the intervals of Z_{k+1} begins where the intervals of Z_k end, G_k is actually comprised of intervals, the largest of which, have length $b^{-k} + b^{-k-1} + b^{-k-2} + ... = \frac{b^{-k}}{1-\frac{1}{b}} = \frac{b^{-k+1}}{b-1}$ and again have centers b^{-k+1} apart. Therefore, we apply a similar argument where Bob aims to contain his intervals in infinitely many G_k's rather than Z_k's.

Theorem 3.3. *If* $\alpha > \frac{(b-1)\beta+\beta-1}{2(b-1)\beta-(b-1)\beta^2-\beta+\beta^2}$*, then* $F_{b,0}$ *is* (α, β)*-losing.*

Proof. Let $(\alpha, \beta) \in I$ such that $\alpha > \frac{(b-1)\beta+\beta-1}{2(b-1)\beta-(b-1)\beta^2-\beta+\beta^2}$. Then, since $\frac{b-1}{b}\left(1 - \frac{2\alpha(1-\beta)}{1-\alpha\beta}\right) < \frac{1-\beta}{b\beta}$, there are infinitely many ordered pairs (n, k) such that

$$\frac{b-1}{b}\left(1 - \frac{2\alpha(1-\beta)}{1-\alpha\beta}\right) < \frac{b^{-k}}{d(\alpha\beta)^n} \leq \frac{1-\beta}{b\beta}.$$

(If $\log_b \alpha\beta$ is rational, proper choice of d is necessary.) Then, for each pair, Bob can ensure that $B_{n+m_n} \subset G_k$ for some m_n. As in the previous proof, he can center B_n around the center of one of the largest intervals in G_k. Also, he can ensure that $B_{n+m_n} \subset \mathbf{B}(c, d(\alpha\beta)^n - \frac{2d(\alpha\beta)^n\alpha(1-\beta)(1-(\alpha\beta)^{m_n})}{1-\alpha\beta}) \subset G_k$ (the largest intervals of which are of size $\frac{b^{-k+1}}{b-1}$). Therefore, Bob can ensure that the point of convergence is in infinitely many G_k's and therefore not in $F_{b,0}$. □

4 Freiling's Set

In [Fr], Chris Freiling considered the set of numbers with a 0 or 5 somewhere in their base 6 decimal expansion, or in our terminology, the set C_6. This set is notable because it is $(\frac{1}{2}, \frac{1}{2})$-winning but $(\frac{1}{3}, \frac{1}{2})$-losing (which is counterintuitive), as Freiling proved. In [Dr], Vladmir Dremov considered a similar set, C_4, and found that it was $(\frac{1}{2}, \frac{1}{2})$-losing, but winning in a small region around $(\frac{1}{2}, \frac{1}{2})$. In this section, we will attempt to further examine why this family of sets exhibits this nature by studying more of its Schmidt diagram. We will apply a similar type of strategy we used to show $F_{b,0}$ was losing, except we apply the ideas to winning arguments for C_b. First, though we will begin by stating the results from [Fr, Dr].

Theorem 4.1. *(1) C_6 is $(\frac{1}{2}, \frac{1}{2})$-winning but $(\frac{1}{2}, \frac{1}{3})$-losing.*

(2) C_4 is $(\frac{1}{2}, \frac{1}{2})$-losing but (α, β)-winning for $\max\left(\left|\frac{1}{2} - \alpha\right|, \left|\frac{1}{2} - \beta\right|\right) < \epsilon, 0 < |\ln(4\alpha\beta)| \leq \epsilon$, *where* $\epsilon = 2^{-12}$.

Proof. (1) See [Fr].

(2) See [Dr].

□

Now we will expand on the losing arguments that Freiling and Dremov used to give more losing points for C_b.

Theorem 4.2. *If $\alpha\beta = \frac{1}{b}$ and $\alpha \geq \frac{2}{b}$, then C_b is (α, β)-losing. Additionally, it is $(\alpha, \beta(\frac{1}{b})^n)$-losing for all $n \in \mathbb{N}$.*

Proof. Let $(\alpha, \beta) \in I$ such that $\alpha\beta = \frac{1}{b}$ and $\alpha \geq \frac{2}{b}$. For the (α, β)-game, Bob first chooses the interval $[0, 1]$. Since $\alpha \geq \frac{2}{b}$, A_1 must include at least one of the complete intervals $[\frac{1}{b}, \frac{2}{b}]$, $[\frac{2}{b}, \frac{3}{b}]$, $[\frac{3}{b}, \frac{4}{b}]$, $\cdots$, $[\frac{b-2}{b}, \frac{b-1}{b}]$. Therefore, since $\alpha\beta = \frac{1}{b}$, Bob can then choose one of these intervals for B_1

and not have a 0 or $b-1$ in the first decimal place, and he may repeat this strategy to ensure that every decimal place is not a 0 or $b-1$.

For the $(\alpha, \beta(\frac{1}{b})^n)$-game, Bob may similarly begin by choosing $[0,1]$, and then as B_k simply choose one of the intervals that he may have chosen as $B_{k(n+1)}$ in the original (α, β) game. By this strategy, he can also ensure that C is also $(\alpha, \beta(\frac{1}{b})^n)$-losing for all $n \in \mathbb{N}$. □

Now, we will consider winning points for C_b. Let us define

$$V_k := \{x \text{ with } 0 \text{ or } b-1 \text{ at its } k^{\text{th}}\ b\text{-ary place or on}\}.$$

Then, we may again note that V_k is made up of intervals, the largest of which will be of length $\frac{2}{b-1} \cdot b^{-k+1}$ whose centers are b^{-k+1} apart. For instance, V_1 contains the intervals $(-\frac{1}{b-1}, \frac{1}{b-1})$ and $(\frac{b-2}{b-1}, \frac{b}{b-1})$. Then, if Alice can contain one of her intervals in one V_k, then the point of convergence will be in C_b. We will then use a similar strategy as the one used in our losing arguments for $F_{b,w}$ to show how Alice can win on C_b. One significant difference between our winning arguments for C_b and our losing arguments for $F_{b,w}$ is that here Alice cannot choose d. When $\log_b \alpha\beta$ is rational, this will make it more difficult for Alice. First, though we will just consider when $\log_b \alpha\beta$ is irrational, where it does not make a difference (since all values between the bounds will be achieved regardless of d, meaning we only need the upper bound to be greater than the lower bound).

Theorem 4.3. *If* $\beta > \frac{(b+1)\alpha-2}{\alpha((2b-4)-(b-3)\alpha)}$ *and* $\log_b \alpha\beta \notin \mathbb{Q}$, *then* C_b *is* (α, β)*-winning.*

Proof. Let $(\alpha, \beta) \in I$ such that $\beta > \frac{(b+1)\alpha-2}{\alpha((2b-4)-(b-3)\alpha)}$. Then, since $1-\alpha > \frac{(b-1)\alpha}{2} - \frac{(b-1)\alpha\beta(1-\alpha)}{1-\alpha\beta}$, there is some order pair (n, k) such that

$$\frac{(b-1)\alpha}{2} - \frac{(b-1)\alpha\beta(1-\alpha)}{1-\alpha\beta} < \frac{b^{-k+1}}{d(\alpha\beta)^n} < 1-\alpha.$$

Then Alice can ensure $A_{n+m} \subset V_k$ for some m, since she may center A_n around a V_k interval, as $d(\alpha\beta)^n(1-\alpha) > b^{-k+1}$ and she may ensure $A_{n+m} \subset \mathbf{B}\left(c, d\alpha(\alpha\beta)^n - \frac{2d(\alpha\beta)^{n+1}(1-\alpha)(1-(\alpha\beta)^m)}{1-\alpha\beta}\right) \subset V_k$ since $d\alpha(\alpha\beta)^n - \frac{2d(\alpha\beta)^{n+1}(1-\alpha)}{1-\alpha\beta} < \frac{2}{b-1} \cdot b^{-k+1}$. Thus Alice may ensure that the point of convergence is in some V_k and thus in C_b. □

Now we consider the case of $\log_b \alpha\beta \in \mathbb{Q}$, which is more complicated. We will take two approaches to this case. Firstly, we will approach it as we did with the winning argument for $F_{b,w}$, where we will set the ratio between the upper and lower bound to be greater than $\frac{1}{\alpha\beta}$, ensuring there exists an (n,k) pair that satisfies the inequality for every k (even though only one is needed). The second approach we will take is to use a theorem from [Ke], which states that the region between two parallel lines of rational slope $\frac{p}{q}$ and width greater than $\frac{1}{\sqrt{p^2+q^2}}$ (and thus vertical distance greater than $\frac{1}{q}$) must include a lattice point. Then, by taking the negative logarithm of our inequality and interpretting it as a region between two parallel lines, we can see that the ratio between the upper and lower bounds must be greater than $b^{\frac{1}{q}}$ where $\log_b \alpha\beta = \frac{p}{q}$ is the slope of these lines. This brings us to our final theorem.

Theorem 4.4. *If* $\log_b \alpha\beta = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{Z}, p, q$ *relatively prime, and* $\dfrac{1-\alpha}{\frac{(b-1)\alpha}{2} - \frac{(b-1)\alpha\beta(1-\alpha)}{1-\alpha\beta}} > \frac{1}{\alpha\beta}$ *or* $\dfrac{1-\alpha}{\frac{(b-1)\alpha}{2} - \frac{(b-1)\alpha\beta(1-\alpha)}{1-\alpha\beta}} > b^{\frac{1}{q}}$*, then* C_b *is* (α,β)*-winning.*

Proof. Let $c_1 = \frac{(b-1)\alpha}{2} - \frac{(b-1)\alpha\beta(1-\alpha)}{1-\alpha\beta}$ and $c_2 = 1-\alpha$. Let $(\alpha,\beta) \in I$ such that $\frac{c_2}{c_1} > \frac{1}{\alpha\beta}$. Then, there is some pair (n,k) such that

$$c_1 < \frac{b^{-k+1}}{d(\alpha\beta)^n} < c_2.$$

As in the previous theorem, then Alice can ensure $A_{n+m} \subset V_k$ for some m, and thus C_b is (α,β)-winning.

Alternatively, let $(\alpha,\beta) \in I$ such that $\frac{c_2}{c_1} > b^{\frac{1}{q}}$. Then, the region in the coordinate plane (with n and k as variables)

$$-\log_b c_1 - \log_b d - n\log_b \alpha\beta + 1 > k > -\log_b c_2 - \log_b d - n\log_b \alpha\beta + 1$$

has slope $\log_b \alpha\beta = \frac{p}{q}$ and vertical distance greater than $\frac{1}{q}$ (and thus width greater than $\frac{1}{\sqrt{p^2+q^2}}$). By [Ke], this region contains a lattice point (n,k), and thus that latttice point must satisfy

$$c_1 < \frac{b^{-k+1}}{d(\alpha\beta)^n} < c_2.$$

Again, then Alice can ensure $A_{n+m} \subset V_k$ for some m, meaning C_b is again (α,β)-winning. □

This last theorem in fact explains the phenomenon which Freiling and Dremov observed with this family of sets. Alice's ability to win is not only associated with how small α is and how large β is, but also with how large the denominator of $\log_b \alpha\beta$ is (if it is even rational). Both of Freiling and Dremov's losing results are implied by Theorem 4.2. Freiling's winning result is implied by Theorem 4.3, since $\log_6 \frac{1}{4}$ is irrational and $(\frac{1}{2}, \frac{1}{2})$ lies in the described region. Dremov's winning result is implied by Theorem 4.4. By observation, it can be seen that the region which Dremov describes lies nowhere near the curves $\log_4 \alpha\beta = -\frac{1}{2}$ and $\log_4 \alpha\beta = -\frac{3}{2}$ (and Dremov excludes the points on the line $\log_4 \alpha\beta = -1$), so any points in the region for which $\log_4 \alpha\beta$ is rational, this value must have denominator three or greater. Again, by observation, we can see that Dremov's region lies entirely within the region described in Theorem 4.4 for $q = 3$. Thus, the theorems which we have presented in this section imply and explain the results of Freiling and Dremov regarding this family of sets.

5 Conclusion and Future Research

In our research, we determined the winning and losing values for vast portions of the global Schmidt diagram for the set $F_{b,w}$, as well as for C_b. We made significant progress in determining the Schmidt Diagrams for these sets and we provided an idea of the boundary of the winning and losing zones for our sets. Additionally, we invented new, abstract methods for proving points in I to be winning or losing. Our ideas culminated in winning and losing strategies in which Alice or Bob could have complex strategies taking m moves in order to achieve their goals. Our research improved immensely the daunting prospect of finding a complete, non-trivial Schmidt Diagram.

Clearly, the most significant future research connected with our project would be to find a complete, non-trivial Schmidt Diagram. As of now, there are only four known Schmidt Diagrams, so any additional complete diagram would be important. More specifically, the complete Schmidt Diagram for the set $F_{b,w}$ or at least for one of these types of sets and a fixed b would be particularly interesting. Additionally, we found that there were points that were losing only when $w = 0$ or $w = b - 1$, and it would be interesting to investigate the claim that $F_{b,x} = F_{b,y}$ for $x, y \in \mathbb{Z}$ and $0 \leq x, y \leq b - 1$.

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