GAME THEORY

JENNIFER YUAN AND SHREYA SINHA

ABSTRACT. This paper is an exploration of game theory. We will focus on *combinatorial games* and the strategies players may use within the game. In section 2, we will define concepts essential to our study of combinatorial games. We will then explore *normal-play games* and broad generalizations we can make of all such games in section 3. In section 4, we explore a game called Nim and the Sprauge-Grundy theorem, which allows us to explore an important property of *impartial games*.

1. Combinatorial Games

1.1. Combinatorial Games.

Definition 1.1. A *combinatorial game* is a game played between 2-players, Louise and Richard. The game consists of the following:

(1) A set of possible positions. These are the states of the game (e.g. the starting board of a chess game).
(2) A move rule indicating what positions Louise can move to and what positions Richard can move to on their turn.
(3) A win rule indicating a set of terminal positions where the game ends. Each terminal position has an associated outcome, either Louise wins and Richard loses (denoted +-), Louise loses and Richard wins (-+), or the game is a draw (00).

To play a combinatorial game, a starting position is chosen with a designated player making the first move. The starting position indicates how the game will look like before it is played. Then, the two players alternate in making moves until a terminal position is reached, indicating the end of the game. A terminal position is reached when one of the two players has won the game or when no possible moves can be made and the game ends with a draw.

Combinatorial games may range from simple games such as Tic-Tac-Toe to more complicated ones such as Chess. However, these games have no element of randomness, so games such as Monopoly that require the use of a dice or a spinner are not considered to be combinatorial games. Combinatorial games thus rely on the player's strategy and moves to win, whereas games with probabilistic elements depends on chance and not wholly in the player's strategy.

Example. A common combinatorial game is Pick-Up-Bricks. In this game, a pile of bricks is placed in the center. Each move consists of a player removing 1 or 2 bricks from the pile. The game ends when the pile is empty, and the last player to take a brick wins.

1.2. **Game Trees.** A *game tree* is a helpful tool to diagram combinatorial games. A *branch* is a line that connects two nodes, and each branch represents a player's move. The point that branches connect are called *nodes*, and each models a new state of the game. *Terminal nodes* - ones that only have one branch connecting it to another node - indicate the possible outcomes.

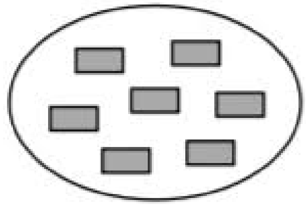

Figure 1. Pick-Up-Bricks

The topmost node is called the *root node*, and represents the beginning state of the game. Each branch node is annotated with an L or an R to indicate whose turn it is to play, with L for Louise and R for Richard.

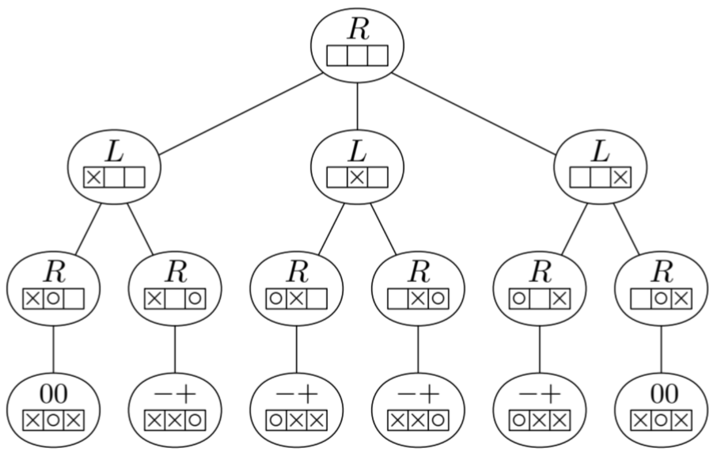

Figure 2. Game Tree Of A Tic-Tac-Toe Game Row

The *depth* of a game tree is defined as the maximum number of possible moves from the start to the end of the game (i.e. the longest path from the root node to a terminal node).

The path that leads to decisive victory for a player may be called a *winning strategy*, and ones that lead to a draw outcome is called a *drawing strategy*.s

The power of game tree lies in the fact that it allows us to diagram all possible combinatorial games regardless of how they are played or any other detail specific to them. This allows us to explore broad properties of games, as we will see in the next section.

1.3. **Zermelo's Theorem.** An important theorem relating to combinatorial games is Zermelo's Theorem, which states that in all such games, either Louise has a winning strategy, Richard has a winning strategy, or both players have a strategy that guarantees a draw. There are no combinatorial games in which one player doesn't have a winning strategy or the game cannot end in a draw definitively.

The power of this theorem is not very intuitive, because it means there exists a winning or drawing strategy in games like chess, where people play professionally. Of course, the reason chess is still played is because there are so many possible positions a player can reach within the game that we have no way of documenting all of them, and no way of finding that strategy at the moment.

Theorem 1.2 (Zermelo's Theorem)**.** In all combinatorial games, at exactly one player has a winning strategy or both players have a draw strategy.

Proof of Zermelo's Theorem. We will prove this by induction. For the base case of depth 0, the game's outcome is already decided as a win for Louise (+-), a win for Richard (-+), or a draw (00).

For the inductive step, assume for a tree of depth $n > 0$ that this assumption holds for all trees with smaller depth. Let $N_1, N_2, \dots N_\ell$ denote the nodes that the starting player can reach in one move. For $1 \leq i \leq \ell$ let T_i be the subtree with root node N_i. Each game T_i where $1 \leq i \leq \ell$ must thus be type +-, -+,or 00 as per the assumption. We can thus label each game path as strategies $\mathcal{L}_i$ for Louise and $\mathcal{R}_i$ for Richard such that either one of these is winning or both are drawing. A strategy $\mathcal{L}$ thus exists for Louise by combining all individual strategies $\mathcal{L}_1, ..., \mathcal{L}_\ell$, and the same for Richard denoted $\mathcal{R}$ by combining $\mathcal{R}$. Assume Richard is the first to move. We then must consider the following cases:

(1) Case 1: At least one of $T_1, T_2, ..., T_\ell$ is type -+. Let T_i be type -+; Richard thus has a strategy $\mathcal{R}_i$ from position N_i, so the winning strategy is to have Richard move to position N_i from the starting position R.
(2) Case 2: All of $T_1, ..., T_\ell$ are type -+. Here, every $\mathcal{L}_1, ..., \mathcal{L}_\ell$ is a winning strategy for Louise, so $\mathcal{L}$ is a winning strategy for her.
(3) Case 3: None of $T_1, ..., T_\ell$ is -+, but at least one T_i is 00. This means that Richard's best strategy is $\mathcal{R}_i$ because he will not lose, and draws instead. Thus, $\mathcal{R}$ is a drawing strategy and Richard must move to position N_i to play by it. In addition, each $\mathcal{L}_1, ..., \mathcal{L}_\ell$ must then be a drawing or winning strategy, so $\mathcal{L}$ is a drawing strategy for Louise.

When Louise is the first to move, the parallel argument holds. □

2. Normal-Play Games

Some combinatorial games can be classified into more specific types. One such is a *normal play* game, where the win rule indicates that the winner of the game is the last player to make a move (i.e. there cannot be a draw). Some normal play games can be categorized further as *impartial* if the set of moves that either player can make is the same, such as in the Pick-up Bricks game. If a game is not impartial, it is *partizan*, meaning that the player's respective set of moves are not the same. For example, chess is partizan because each player has their respective set of pieces that they may move.

2.1. Positions and Their Types.

In this section, we will define new notation to represent positions, classify and describe the different types of positions, and figure out how to determine the type of a position in order to explore more general properties of games.

Positions are denoted as α, β, and γ, where each notation represents a state of a game. The purpose of position notation is to emphasize the relation between the current state, and the possible future states of the game. Thus, for a possible game, its position notation may look like this: $\gamma = \{\alpha_1, \alpha_2, ..., \alpha_n | \beta_1, \beta_2, ..., \beta_n\}$, where γ is the current state of the game, α_i represents all the possible positions Louise may bring the game to if it her turn using her move rules ($\gamma \to \alpha_i$), and β_i represents all the possible positions Richard may move to ($\gamma \to \beta_i$).

Each position can be classified into specific types. Defining types and their properties will make it easier to predict the outcomes of several simultaneous games at once. There are four potential types of a game:

(1) L : Louise has a winning strategy from this position regardless of whose turn it is.

(2) R: Richard has a winning strategy from this position regardless of whose turn it is.
(3) N: The current player whose turn it is has the winning strategy.
(4) P: The player whose turn it is not has the winning strategy.

This also allows us to analyze what type of position the game is by analyzing what kind of positions Richard and Louise can move to:

	some β_j *is type* R *or* P	*all of* $\beta_1, \dots, \beta_n$ *are types* L *or* N
some α_i *is type* L *or* P	N	L
all of $\alpha_1, \dots, \alpha_m$ *are types* R *or* N	R	P

FIGURE 3. Determining the Current Type Using the Types of Possible Subsequent Positions

In this paper, we will be focused on impartial games. In these games, each position is always type N or P.

2.2. **Sum of Positions.** In a game of Pick-Up Bricks, the pile of bricks may be broken down into many different-sized piles over the course of the game. If we call each piece a component of the game, then we can view a Pick-Up Bricks position as a sum of these components.

Definition 2.1. If α and β are positions in normal-play games, then we define α + β as the starting position in a combined game where game α and game β are played simultaneously. To move in the game α + β, a player chooses to make a move in either game α or game β. For example, if a player makes a move in game α, then the player is moving from α to α' so the new position of the game may be described as α' + β. Similarly, if a player makes a move in game β, then the player is moving from β to β' so the new position may be described as α + β'.

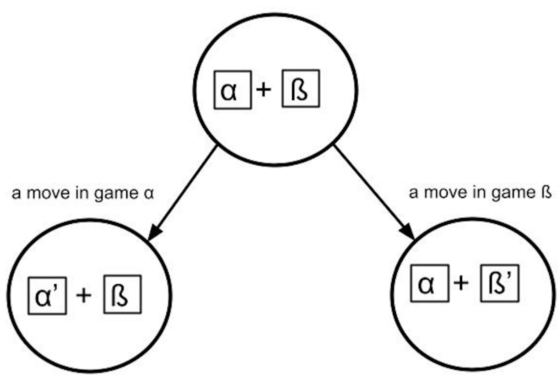

FIGURE 4. Example Game Tree of Combined Game $\alpha + \beta$

If β is type P, then α and α + β are the same type. To understand the reasoning behind this, let's look at an example. Suppose position β is type P and position α is type L. In a case where it is Louise's turn, she can start the game by making a move in game α. If Richard also makes a move in game α, then Louise can continue to make a move in game α. If Richard makes a move in game β, then Louise will proceed to make a move in game β as well. Since game α is type L, Richard will eventually run out of moves in game α and will be forced to make the first move in game β. This means that Louise will always have a strategy to win game β because she will always be the player moving second in the game making $\alpha + \beta$ also type L. The same strategy can be followed for a game where Richard makes the first move.

If α and β are both type L, then $\alpha + \beta$ is also type L. Similarly, if α and β are both type R, then $\alpha + \beta$ is also type R. The reason for this is because if both game α and β were type L or type R, it means that no matter what the other person does in either of the two games, he or she will still not be able to win either game α or game β. Therefore, $\alpha + \beta$ would be of the same type.

+	L	R	N	P
L	L	?	?	L
R	?	R	?	R
N	?	?	?	N
P	L	R	N	P

FIGURE 5. Types of sum

How do we determine the type of a game? Looking at a game of Domineering may help. Domineering is a normal-play game played using some squares from a rectangle array. On Louise's turn, she may place a 2 x 1 domino over two unoccupied squares. On Richard's turn, he may place a 1 x 2 domino over two unoccupied squares. Since this is a normal-play game, the last player to move wins.

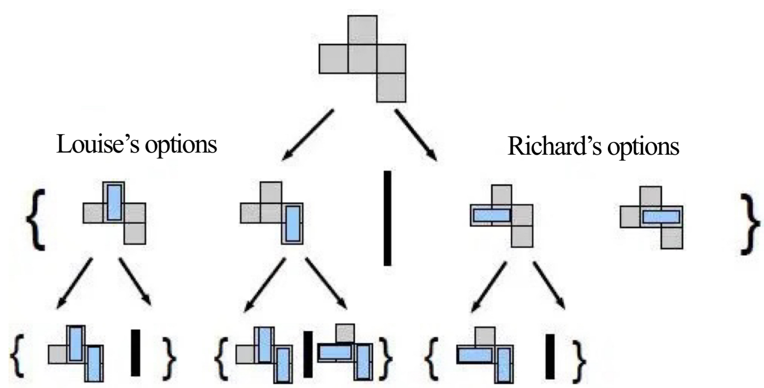

FIGURE 6. Domineering

A easy way to look at a game of domineering is to remove any unusable components after a player makes a move. For example, if Louise was the first player and chooses the first option as shown in Figure 6, then the two squares in the center will be removed. The game would be left with one 1 x 1 piece and one 2 x 1 piece. Since no move can be made with the 1 x 1 piece, the piece may be removed as well. The result would be a new position with only one 2 x 1 piece remaining. The remaining 2 x 1 piece is a type L, meaning that Louise will win the game.

2.3. **Equivalence of Positions.**

Definition 2.2 (Equivalence)**.** We say that position α and β in (possibly two different) normal-play games are *equivalent* if they have the same type for any position γ added. In other words, if $\alpha + \gamma$ and $\beta + \gamma$ has the same type for any position of γ, then $\alpha \equiv \beta$

There are some fundamental properties of equivalence that are useful to know. If α, β, and γ are positions in normal-play games, then:

(1) $\alpha \equiv \alpha$ (reflexivity),
(2) $\alpha \equiv \beta$ implies that $\beta \equiv \alpha$ (symmetry),
(3) $\alpha \equiv \beta$ and $\beta \equiv \gamma$ implies that $\alpha \equiv \gamma$ (transitivity)
(4) $\alpha + \beta \equiv \beta + \alpha$ (communtativity)
(5) $(\alpha + \beta) + \gamma \equiv \alpha + (\beta + \gamma)$ (associativity)

It is also important to note that if $\alpha \equiv \alpha$', then α and α' have the same type.

$$\alpha \xleftrightarrow[\text{type}]{\text{same}} \alpha + \beta \xleftrightarrow[\text{type}]{\text{same}} \alpha' + \beta \xleftrightarrow[\text{type}]{\text{same}} \alpha'$$

However, when two positions have the same type, it does not mean that the two positions are equivalent.

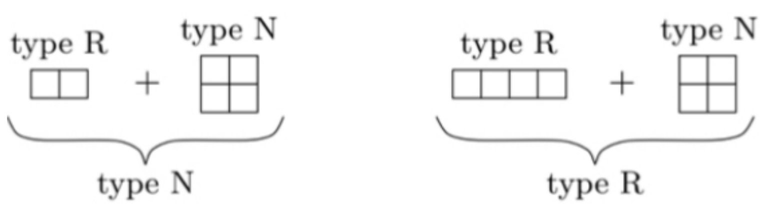

Figure 7. Two sums in Domineering with different types

When determining if two (or more) positions are equivalent, type P serves as the number zero. In other words, if β is type P, then $\alpha + \beta \equiv \alpha$. The reasoning behind this is explained in the sum of positions. Any position added with a position of type P will result in the sum of the two positions to be the same as the first position. So, if β is type P, then $\alpha + \beta$ is the same type as α. Two positions are equivalent when they have the same type when a third position is added, so $\alpha + \beta \equiv \alpha$.

3. Nim and the Sprauge-Grundy Theorem

Nim is an impartial game in which positions consist of ℓ piles of stones sizes $\alpha_1, \alpha_2, ..., \alpha_\ell$. A player moves by choosing a pile and removing any positive number of stones from that pile. The winner is the person who picked up the last stone.

The notation $*a$ represents a Nim position consisting of one pile of a stones. We call $*a$ a *nimber*. A game with piles of sizes $a_1, a_2, ...a_\ell$ can be written as $*a_1 + *a_2 + ... + *a_\ell$.

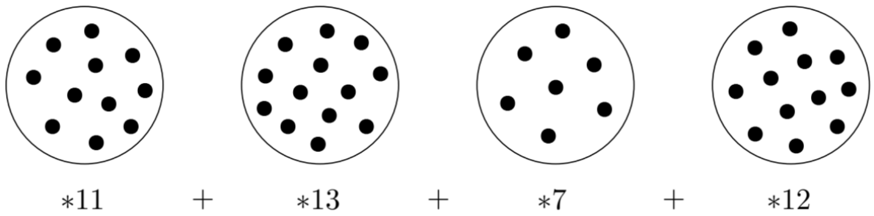

Figure 8. A Position in Nim

Each pile in a game of Nim can be broken into subpiles based on the binary expansion of the number of stones in the pile.

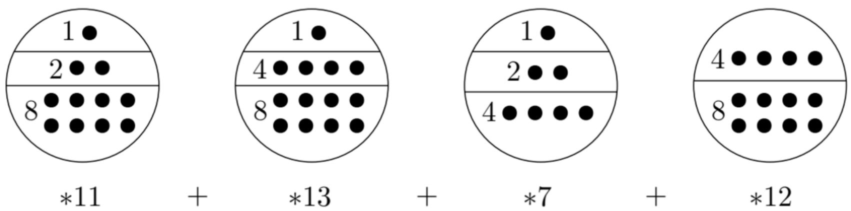

Figure 9. Subgroups Based on Binary Expansion

For example, in the figure above, $*11$ is broken into subgroups of size 1, 2, and 8 because the binary expansion of 11 is 1+2+8 = 11. This allows us to introduce a new property of the game Nim: *balance*. Each position $*a_1 + *a_2, ... + *a_\ell$ is balanced if, for every power of 2, the total number of subgroups of that size across the entire game is even. If this is not true of the position, then it is an *unbalanced* position.

The ability to get from an unbalanced position to a balanced one allows us to find a winning strategy.

Theorem 3.1 (Balancing Nim)**.** A player who moves on an unbalanced position can always move it back to a balanced one.

Proof. Consider an unbalanced game $*a_1 + *a_2, ... + *a_\ell$, and assume that 2^m is the largest power of 2 for which there are an odd number of subgroups. Suppose that a pile $*a_i$ is a pile with a subpile size 2^m. Pick up all the subgroups of $*a_i$ that are less than or equal to 2^m in size. This will be at most 2^m stones, because in the worst case of picking up piles of all sizes less than 2^m, that is still $2^0 + 2^1 + ... + 2^{m-1} = 2^m - 1$ stones. For all the subpiles of size 2^j such that $j < m$, if the number of subpiles is now odd, put the 2^j stones initially picked up back in $*a_i$ to leave it even. This leaves the final position balanced. □

Proposition 3.2 (Types of Nim Game Positions)**.** Another important property of Nim games is that each balanced Nim position is type P and each unbalanced Nim position is type N.

Proof. Assume a balanced game $*a_1 + *a_2, ... + *a_\ell$. Any move the player makes leaves the game unbalanced, because that player only has access to one pile, and thus only one subpile of the sizes in that pile. The second player may use the balancing procedure to re-balance the game. This cycle continues, and the end position of the game - one where there are no more

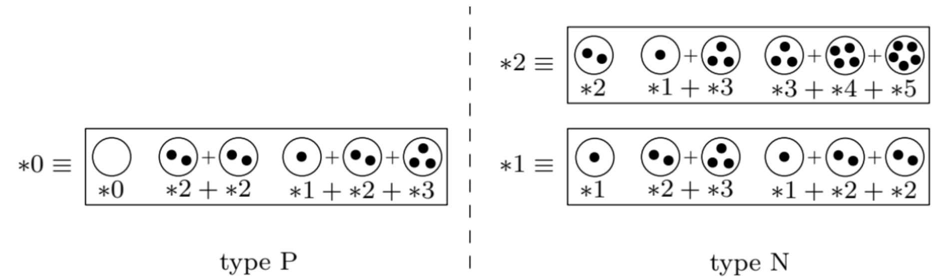

Figure 10. Position Types of Nim

stones - is a balanced one, so the second player has the winning strategy in this case, and it is type P. If the game starts unbalanced, the first player has the ability to balance it and thus has a winning strategy, making it type N. □

The idea of balancing the game also allows us to think about the idea of equivalent positions in the game. As we found earlier, any type P positions of a game are equivalent. Since $*0$ is type P, it follows that any balanced game is equivalent to $*0$. This allows us to simplify games down to single nimbers.

After learning that every Nim position is equivalent to a nimber, it is actually true that every position in any impartial game is equivalent to a nimber! This is known as the Sprague-Grundy Theorem.

To begin, let's define what MEX is.

Definition 3.3 (MEX Value)**.** For a set S = $\{\alpha_1,\ \alpha_2,\ ...,\ \alpha_n\}$ of nonnegative integers, we define the *Minimal Excluded value*, abbreviated MEX, of S to be the smallest nonnegative integer b which is not one of $\{\alpha_1 ... \alpha_n\}$. For instance, the MEX of the set {0,1,2,5,8} is 3.

This leads us to the MEX Principle:

Theorem 3.4 (MEX Theorem)**.** For a position $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_k\}$, if $\alpha_i \equiv *a_i$ for each $1 \leq i \leq k$, then $\alpha \equiv *b$ when b is the MEX of $\{a_1, a_2, ..., a_k\}$.

Proof. Assume a combined position of $\alpha + *b$.

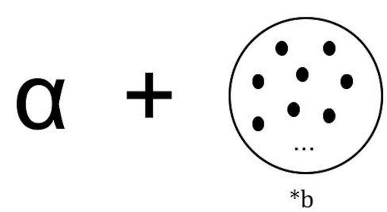

Figure 11. Visualization of the Combined Positions α and *b

Suppose the first player moves $*b$ to $*b'$ where $b' < b$. By definition of a MEX value b, there must be some $a_i = b'$ where $1 \leq i \leq k$. The second player can change α to α_i so that the position of the entire game is $*b' + \alpha_i \equiv *b' + *a_i \equiv 0$. Thus, the second player has a winning strategy.

If, however, the first player moves α to α_j, then because $\alpha_j \neq b$ (by the definition of a MEX value), this position is equivalent to $*(b \oplus \alpha_j)$ where $*(b \oplus \alpha_j) \not\equiv *0$. The second player may then change $*b$ to $*b'$ such that $b' = a_j$, so that $*b' + \alpha_j \equiv *b' + *a_j \equiv *0$. The second player again has a winning strategy, so $\alpha + *b$ must be type P.

Then, because $\alpha + *b$ is type P, $\alpha + *b \equiv *0$. Adding $*b$ to both sides gets $\alpha + *b + *b \equiv *b$, and because $*b + *b$ must be equivalent to $*0$, $\alpha \equiv *b$. □

We can use this to prove the Sprauge-Grundy theorem by induction based on the depth of a game.

Theorem 3.5 (Sprauge-Grundy Theorem)**.** Every position in any impartial game is equivalent to a nimber.

Proof of the Sprauge-Grundy Theorem. Assume α to be a impartial game position. The base case is when α has depth $d = 0$, so $\alpha \equiv *0$. For the inductive step, assume $d > 0$ and the theorem holds for all positions with depth less than d. Since $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_\ell\}$, then each α_i has depth less than d and is equivalent to some $*a_i$. Thus, $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_\ell\} \equiv *a_1, *a_2, ..., *a_\ell$. Assume b to be the MEX of $\{a_1, a_2, ..., a_\ell\}$. Then $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_\ell\} \equiv \{*a_1, *a_2, ..., *a_\ell\} \equiv *b$. □

References

[1] DeVos, M., Kent, D. *Game Theory: A Playful Introduction.* American Mathematical Society, 2016.