Enumerative Combinatoric Paper

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1 Introduction

Enumerative combinatorics mainly collected from Euler rule and other various binomial theorem. It is an important subject in research area in mathematics mostly due to it complexity and the connection with other research topic. In facts, enumerative problems is so complex that many mathematician cannot solve using modern technology. The basic of enumerative combinatorics is counting number of finite sets. In this paper, we going to talk about the fundamental of enumerative combinatorics which is binomial theorem and other related topic. Before we talk about binomial theorem, we need to understand the commonly mistake people usually make and the correct way to do it.

2 Commonly mistake on binomial theorem

Definition 2.1 (incorrect way)**.** When we talk about the binomial theorem, people tend to make a commonly mistake. For example: $(a + b)^2$, People will tend to distribute the square into the equation, and they got $a^2 + b^2$. Which is an incorrect way to do it.

Definition 2.2 (correct way)**.** The correct way to do binomial theorem is either the box method or the foil method.

Foil Method

$$(a + b)(a + b) = a^2 + 2ab + b^2\}$$

Box Method

	a	+b
a	a^2	ab
+b	ab	b^2

Figure 1: Box method.

Definition 2.3 (Issue with box method and Foil method)**.** As you can see the foil method and box method take very long time to solve just a simple problem like $(a+b)^2$. Thus, people came up with binomial theorem equation to help them solve problem that are more challenging. This would help them to save times while still get a correct answer

3 The binomial theorem

Theorem 3.1 (binomial theorem)**.** *Binomial theorem equation would help us to solve complex binomial equation faster while still getting the correct answer.*

Let n be a positive integer. For all real numbers x and y, we have

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

Just a reminder that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Proof. Consider the product of n sums, $(x+y)(x+y)\cdots(x+y)$, when computing this product, we take one summand from each parentheses, multiply them together, then repeat this in all 2^n possible ways and add the results. We get a product equal to $x^k y^{n-k}$ each time we take k summands equal to x. There are $\binom{n}{k}$ times, and the proof is complete

Took from: Miklos Bona- a walk through combinatorics, an introduction to Enumeration and Graph Theory □

Example 1. Expand $(x+y)^5$

$$\binom{5}{0}x^5y^0 + \binom{5}{1}x^4y^1 + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}x^1y^4 + \binom{5}{5}x^0y^5$$

$$\binom{5}{0} = \frac{5!}{0! * 5!} = 1$$

$$Answer\ :\ x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1y^5$$

Example 2. Prove: $0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^k$

Set X=-1 and y=1 in the binomial theorem

$$(-1+1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k(1)^{n-k}$$

$$0 = \sum_{k=0}^{n}\binom{n}{k}(-1)^k$$

4 Pascal triangle

Theorem 4.1 (The pascal triangle)**.** *Help you to calculate the binomial theorem and find combinations way faster and easier.*

We start with 1 at the top and start adding number slowly below the triangular.The number in the pascal triangle is binomial coefficients

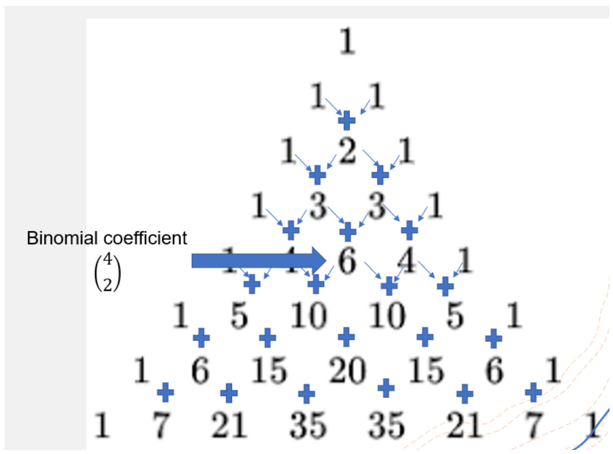

Figure 2: pascal triangle

Proof. Let say n and k are all non negative integers, we got

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

This would give us

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

This is true because $\binom{n}{k}$ are the numbers of ways to choose k elements from n while $\binom{n}{k+1}$ is the numbers of ways to choose $k+1$ elements from n. Thus, we got $\binom{n+1}{k+1}$ as counts number of ways to choose $k+1$ elements from n+1 elements ☐

Proof. Let say n and k are all non negative integers and we got $\binom{n}{k}$ choose $n+1$ as one of $k+1$ elements while $\binom{n}{k+1}$ do not choose n+1. This would lead us to $\binom{n+1}{k+1}$ ☐

Example 1. $(3x-6)^4$

Now let solve this problem by using the pascal's triangle. The binomial coefficient we got from the pascal triangle are

1 4 6 4 1

Thus our answer would be

$$1(3x)^4(-6)^0 + 4(3x)^3(-6)^1 + 6(3x)^2(-6)^2 + 4(3x)^1(-2)^3 + 1(3x)^0(-6)^4$$

Answer: $3x^4 - 648x^3 + 1944x^2 - 96x + 1296$

5 Binomial coefficient equation

Theorem 5.1 (Binomial coefficient equation). *N and k are integers ≥ 0 with n≥k. We got*

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Proof. Proof that k elements of n is $\frac{n!}{k!(n-k)!} = \binom{n}{k}$ with $k \le n$ □

Let say the length of K is from the n elements set. Therefore, n ways of picking first elements, n-1,…,n-k+1. Thus we got

$$n\,(n-1)\,...\,(n-k+1) = \frac{n!}{(n-k)!}$$

We want to find unordered list and identify k!. Thus, number of unordered lists we got are

$$\frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{k!\,(n-k)!}$$

Example 1. Find binomial coefficient of $\binom{9}{2}$

$$\binom{9}{2} = \frac{9!}{2!\,(9-2)!} = 36$$

6 Generalized binomial theorem

Theorem 6.1. *The binomial theorem is only truth when n=0,1,2... Let say n is negative number or factions , we would need to use generalized binomial theorem*

$$(1+b)^r = \sum_{k=0}^{\infty} \binom{r}{k} b^k,\ r \in \mathbb{R}$$

Example 1. *W*hat does **r choose k mean when r is not positive integers** $\binom{r}{k}$

$$\binom{r}{k} = \frac{r\,(r-1)\,(r-2)\,(r-3)\,..\,(r-(k-1))}{k!}$$
$$\binom{-3}{6} = \frac{(-3)\,(-4)\,(-5)\,(-6)\,(-7)\,(-8)}{6*5*4*3*2*1} = 28$$

Example 2. Let say b=-x, we got

$$\sum_{k=0}^{\infty}\binom{-1}{k}(-x)^k = (1-x)^{-1}$$

As we saw from the last two example, generalized binomial theorem can help us solve problem when n is negative number.

7 Trinomial theorem

Theorem 7.1 (Let say i,j,k will be non-negative number, we got)**.**

$$(a+b+c)^n = \sum_{i+j+k}\binom{n}{i,j,k}a^i b^j c^k$$

Just a reminder that $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$

Proof. Let $a, b, c \in \mathbb{R}$, while we know that expansion of trinomial $(a+b+c)^n$, we got

$$(a+b+c)^n = (a+b+c)\,(a+b+c)\,...\,(a+b+c)$$

As we distribute a,b,c we expand x,j as many as we expand y,k. Thus we got $n = i + j + k$. This would give us $\binom{n}{i,j,k}$ for each $n = i + j + k$ while a, b, c are non integers. This would give us

$$(a+b+c)^n = \sum_{i+j+k}\binom{n}{i,j,k}a^i b^j c^k$$

□

Example 1. How many terms in $(a+b+c)^7$ ***?*** *Let list all the given out*

i+j+k=7
I=2
J=0
K=5
Plug in all the given into the equation we got

$$(a+b+c)^7 = \sum_{2+0+5}\binom{7}{2,0,5}a^2 b^0 c^5 = 36$$

8 Multinomial theorem

Theorem 8.1 (Multinomial theorem). *Invented by Willian L Hosch with the purpose to help us to expand $(x_1 + \cdots + x_k)^n$. The theorem based on binomial theorem but more advance and consist of the sum of many terms*

Let say k is a positive integer and n is a non negative integer, we got

$$(x_1 + \cdots + x_k)^n = \sum \frac{n!}{e_1!e_2! \ldots e_k!} x_1^{e1} x_2^{e2} \ldots x_k^{ek}$$

Where: $e_1, e_2 \ldots e_k \geq 0$, e_i is exponent of x_i in a monomial

$$e_1 + \cdots + e_k = n$$

Proof. The left-hand side counts all k-elements subset of [n + m]. The right-hand side counts the same, according to the number of elements chosen from [n]. Indeed, we can first choose i elements from [n] in $\binom{n}{i}$ ways, then choose the remaining $k - i$ elements from the set $\{n + 1,\ n + 2,\ \ldots n + m\}$ in $\binom{m}{k-i}$ ways.

Considering any one row of the Pascal triangle, we note that the binomial coefficients $\binom{n}{0}$, $\binom{n}{1}$, ... seem to increase as k increases, up to the middle of the row, after which they seem to decrease. As the following theorem shows, this is indeed true for all n.

Took from: Miklos Bona- a walk through combinatorics, an introduction to Enumeration and Graph Theory

Example 1. Proof that multinomial theorem will be binomial theorem when k=2

$$(x_1 + x_2)^n = \sum \frac{n!}{e_1!e_2!} x_1^{e1} x_2^{e2}$$

$\mathrm{e}_1 + e_2 = n$
$\mathrm{e}_2 = n - e_1$
Rename !=e_1
We got

$$(x_1 + x_2)^n = \sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!} x_1^i x_2^{n-i}$$

9 Vandermonde's identity

Theorem 9.1. *m,n, and k are non-negative integer with k≤min (m,n)*

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} \quad r, m, n \in \mathbb{N}_0$$

Figure 3: Proof

Proof. Let say that we have two circle one being label as X and other being label as Y, we got

$|X| = n$
$|Y| = m$
$|X \cap Y| = |X| + |Y| - 0$
$|X \cup Y| = |X| + |Y| = m + n$
This shows
One side: $\binom{m+n}{r}$
Other side: let $0 \leq k \leq r$
Choose K elements from X $\binom{n}{k}$
Choose r-k elemts from Y $\binom{m}{r-k}$
We get $\sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k}$

Example 1. Let m=n=k

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-j}$$

This would give us
$\sum_{i=0}^{n} \binom{n}{i}^2$
Answer: $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$ □

□

10 Binomial theorem general version

Theorem 10.1. *Let say m must be any real number, x must be any real number that* $|x| < 1$ *and sum taken all non-negative integer n, we got*

$$(1+x)^m = \sum_{n \geq 0} \binom{m}{n} x^n$$

Proof. Let use the sum over k of r-kt choose k by r over r-kt by z^k, we got

$\sum_n \binom{a-nt}{k} \frac{a}{a-nt} z^n = x^a$
Where z=x$^{t+1} - x^t andx = 1 for z = 0$
Set t=0 and plug in the given into the equation, we would got
$\sum_n \binom{a-n\times 0}{n} \frac{a}{a-n\times 0} z^n = x^a$
This would give us
$\sum_n \binom{a}{n} z^n = (1+z)^a$
Took from Proofwiki

$\square$

Example 1. Find the power series expansion of $\sqrt{1-4x}$
$(1-4x)^{\frac{1}{2}} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4x)^n$
$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}*\frac{-1}{2}*\frac{-3}{2}\dots\frac{-2n+3}{2}}{n!} = (-1)^{n-1} * \frac{(2n-3)}{2^n*n!}$
$\sqrt{1-4x} = 1 - 2x - \sum_{n\geq 2} \frac{2^n*(2n-3)}{n!} * x^n$
$2^n * (2n-3) \frac{}{n!=2*\frac{(2n-2)!}{n!(n-1)!}}$
We got
$\sqrt{1-4x} = 1 - 2x - \frac{2}{n} \sum_{n\geq 2} \binom{2n-2}{n-1} x^n$