PRIMES CIRCLE FINAL PROJECT

SOFIA MROWKA, ELIZABETH ZHONG, AND NATASA ZUPANSKI

1. Introduction

In this paper, we explore both combinatorial and classical game theory (see Definitions 2.1 and 3.1). Combinatorial game theory is the study of games like Chess or Checkers, where two players alternate turns until one wins the game. Classical game theory studies games like Rock Paper Scissors, where each player simultaneously makes a single decision without knowing the decision of the other player. We will focus on the basic definitions and terms of game theory by diving deeper into two original games: Roads and One Four All. In Section 2, we will discuss the combinatorial side of game theory with Roads, and in Section 3, we will discuss the classical side with One Four All.

2. Combinatorial Game Theory

2.1. **Basic Definitions.** In this section, we will begin by introducing formal definitions that will help us transform these game boards into strategies for winning.

Definition 2.1 ([1, Definition 1.5])**.** A *combinatorial game* is a 2-player game played between Louise and Richard. A combinatorial game consists of:

(1) A set of possible *positions* which are the state of the game.
(2) A *move rule* indicating for each position what positions Louise can move to and what positions Richard can move to.
(3) A *win rule* indicating a set of *terminal positions* where the game ends. Each terminal position has an associated *outcome*, either Louise wins and Richard loses, Louise loses and Richard wins, or it is a draw.

Example 2.2. In the game Tic there is a 3 by 1 array. To move, Louise marks an empty square with a $\circ$ and Richard with a $\times$. If either player gets two adjacent squares marked with his or her symbol, then they win. The game Tic is clearly an example of a combinatorial game as it satisfies all three conditions.

The next thing to consider is the condition for winning. Games can be classified by their condition for winning.

Definition 2.3 ([1, p.3])**.** A combinatorial game is called a *normal play game* when the last player to make a move wins.

Example 2.4. Checkers is an example of a normal play game.

All of the combinatorial games played in this paper are normal play games. Games can also be categorized into the types of moves made by each player.

Definition 2.5 ([1, p.26])**.** An *impartial game* is a normal play game in which the available moves for Richard and Louise are always the same.

Example 2.6. Tic is be an example of an impartial game as the available moves for Richard and Louise are the same.

Definition 2.7 ([1, p.26])**.** A *partizan game* is a normal play game in which the available moves for Richard and Louise are not the same.

Example 2.8. An example of a partizan game is Cut the Cake. In Cut the Cake there is a cake which is rectangular and has horizontal and vertical lines running along the cake indicating where the cake can be cut. Richard can only make horizontal cuts and Louise can only make vertical cuts. From each position in Cut the Cake, Richard and Louise have different available moves, so the game is partizan.

Convention 2.9. When considering a partizan game, we call the players Louise and Richard (for left and right). When the game is impartial, we call the players Richard and Roberto (for right and right).

Definition 2.10 ([1, p.7])**.** A *strategy* is a set of decisions indicating which move to make at each position where that player has a choice.

Definition 2.11 ([1, p.7])**.** A *winning strategy* is a strategy with the property that following it guarantees a win.

In order to expedite the process of finding a winning strategy, it is important to classify and label positions.

Definition 2.12. Positions Types:

(1) **Type L**: Louise has a winning strategy no matter who goes first.
(2) **Type R**: Richard has a winning strategy no matter who goes first.
(3) **Type N**: The Next player has a winning strategy.
(4) **Type P**: The Previous player or the second player has a winning strategy.

Definition 2.13 ([1, Definition 2.4])**.** If α and β are positions in a normal-play game, then we define $\alpha + \beta$ to be a *sum of positions.* To move in $\alpha + \beta$, a player chooses one of the components in which to move. So, for instance if a player may move from α to α', then the player may move from $\alpha + \beta$ to $\alpha' + \beta$. Similarly, if a player can move from β to β', then he or she may move from $\alpha + \beta$ to $\alpha + \beta'$

Definition 2.14 ([1, Definition 2.9])**.** Two positions α and α' in normal play games are called *equivalent* when for every position β, the two positions $\alpha + \beta$ and $\alpha' + \beta$ have the same type.

2.2. **Rules for Roads.** In this section we introduce our original normal play combinatorial game, Roads. The game starts with a set of vertices that are connected by edges, or roads. A player can make a move by going down a road and must continue to turn left or right onto other roads until there are no more turns to take. In this way, this game can be partizan or impartial.

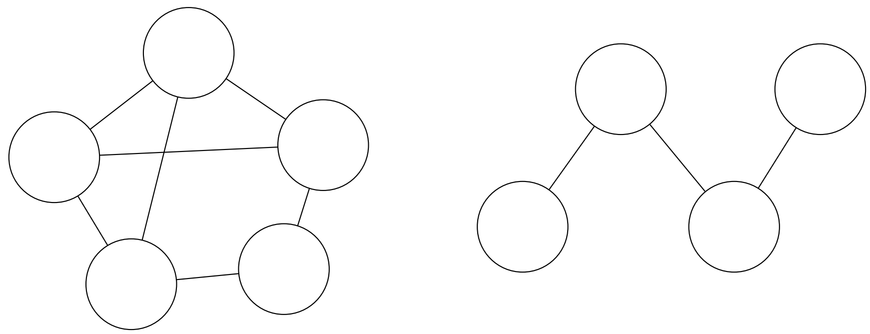


FIGURE 1. Examples of game boards

There are two separate games: Partizan Roads and Impartial Roads. Both games have the same win rule, but the move rules are different.

In Partizan Roads, one player may only make left turns (Louise) while the other player may only make right turns (Richard). While it seems like the other player can make the exact same move by starting where the other player stopped, there is a counterexample of this.

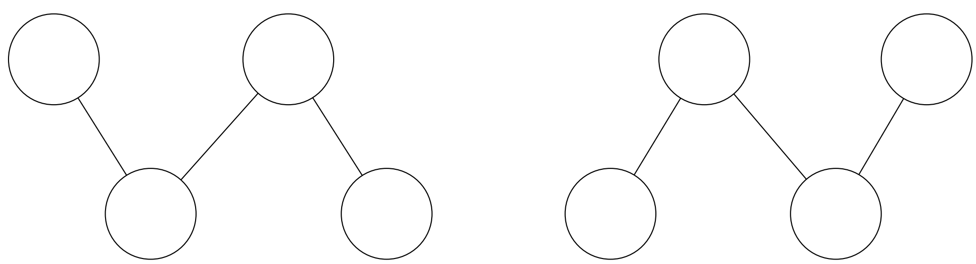


FIGURE 2. Partizan positions in Partizan Roads

Let's look at the board on the left. The move in question is the middle road. In Partizan Roads, Louise can take just the middle road, while Richard must take one of the side roads as well if he is to take the middle road. This is because the game requires that the player continues to turn onto other roads until there are no more turns to take. In the case of Louise, once she takes the middle road, there are no more left turns to make, meaning that she must stop. In the case of Richard, once he takes the middle road, he must take the right turn and go down the next road.

The board on the right is very similar, except that the roles are flipped. On this game board, Richard can take just the middle road but Louise cannot, hence why this version of the game is partizan.

In Impartial Roads, both players can only make right turns (Richard and Roberto). This is clearly impartial, as both players have the same set of moves.

This paper will analyze Impartial Roads between Roberto and Richard, who may only make right turns, as well as Partizan Roads between Louise and Richard, who may make left and right turns, respectively.

2.3. **Roads in Convex Polygons.** We now investigate Roads as played on convex polygons.

Theorem 2.15. *The game Partizan Roads, when played on convex polygons, is impartial.*

Proof. Suppose Richard makes a move from position P_0 at a point A, takes a series of right turns and ends at point B reaching a position P_1. Louise can also make a move at position P_0 but she starts at point B then retraces Richard's moves backwards making a series of left turns and ends at point A with of a position of P_1. Vice versa if Louise starts. Therefore no matter what position Louise or Richards are given to move from they can always end in the same position by retracing each others steps backwards. □

Theorem 2.16. *In Partizan Roads, all positions which are convex polygons are type N.*

Proof. Let's first consider the properties of convex polygons in the context of the partizan version of Roads. Depending on the player, the turns are all left turns or all right turns. This means that both players are able to take all of the roads in a single move. Suppose Louise goes first. Although Louise can only make left turns, there are only left turns to take. Similarly, if Richard goes first, he can always take all of the roads, because they will all be right turns. This means that the player who goes first has a winning strategy, and any convex polygon is type N. □

2.4. **Zig-Zag Roads.** Zig-Zag Roads are a type of position in both Impartial and Partizan Roads. These positions consist of a number of edges/roads, which, when driven down, contain alternating left and right turns. Hence, the position is a familiar "zig-zag" shape.

2.4.1. *Impartial Version.*

Lemma 2.17. *Symmetry: In an impartial game, any position of the form $Z + Z$ is type P.*

Proof. Proof by induction on n, the number of edges in Z. As a base case, suppose $n = 0$. Then the first player loses, so this is type P.

Inductive step: Assume the result holds true for Z with fewer than n edges. Suppose Z is of size n then Roberto makes a move leaving a position $Z' + Z$ and Richard copies that move on Z which leaves the position $Z' + Z'$ which has less then n edges therefore proving that $Z + Z$ is type P. □

Lemma 2.18. *Let Z be a zig-zag with an even number of edges and $Z^\vee$ be it's flip. In an impartial game of roads, Z is equivalent to $Z^\vee$ if and only if Z has an even amount of edges in an impartial game of Roads.*

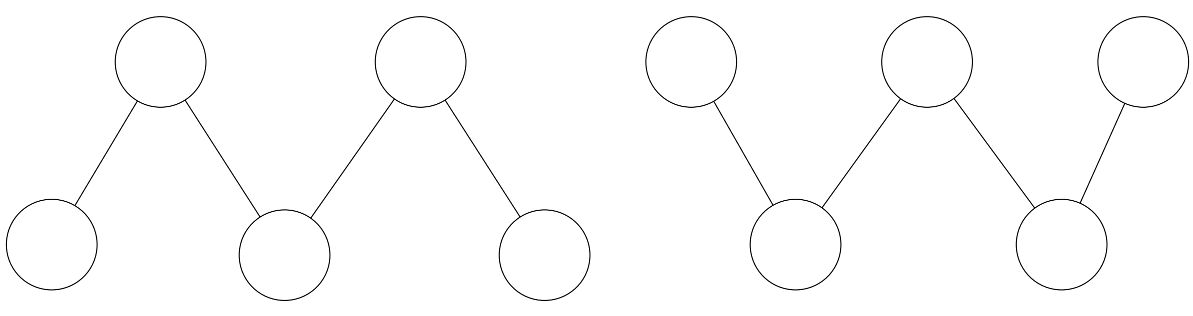


Figure 2: Examples of Z (left) and $Z^\vee$ (right).

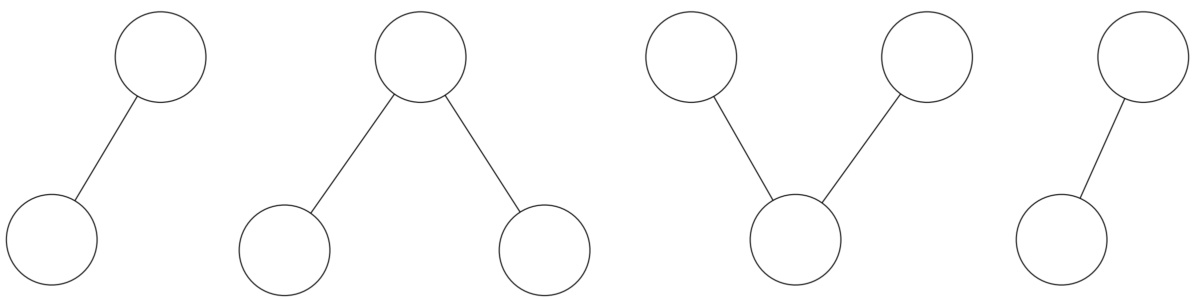
Proof. There are two possible ways to construct $Z^\vee$ from Z: one is to flip Z vertically and the other is to rotate Z around its center vertex by 180 degrees. Considering that a zig-zag with an odd number of edges does not have a center vertex, it cannot, therefore, be rotated in the same manner as an even-edged zig-zag.

As Richard and Roberto are the chosen players for an impartial game of Roads, they can only turn right, and therefore, take any negative sloped edge in a zig-zag without having to turn and take an additional edge. Hence, trying to take a positive sloped edge of a zig-zag will force them to take two edges. Keeping this in mind, suppose that Richard takes the second edge (from left to right) of Z from Figure 2. If we define a flip for an impartial game of Roads to be it's vertical flip, Richard's move lands on the second edge of $Z^\vee$ in Figure 2. However, this edge has a positive slope, so he must, in this case, take at least two edges.

In order for two positions to be equivalent, they must act the same way under summation, meaning that no matter what position you add to them, the result will be the same. This is the case if a player can act the same way on the two positions, which, with the definition of $Z^\vee$ as the vertical flip of Z, does not hold. However, defining $Z^\vee$ as the rotation of Z around it's middlemost vertex by 180 degrees, translates Richards possible moves from Z to $Z^\vee$.

When rotating Richards previous move, taking the second, negative sloped, edge from Z, we find it lands on the third edge of $Z^\vee$. This edge has a negative slope, so Richard can take it without having to make a turn.

To further delve into the rotation of the move, supposing Richard took the second edge of Z from top to bottom, the parallel on $Z^\vee$ would be to take the third edge from bottom to top, and vice versus. This move also results in two equivalent positions, sums a two edged zig-zag and a single edged zig-zag.



Notice that the two single edged zig-zags of Z and $Z^\vee$ are positioned the same way. When a single edge is taken from a larger even edged zig-zag, Z, the result will be the sum of an odd edged zig-zag, α, and an even edged zig-zag, β, if the single edge is not at either end of the zig-zag. Together they are $\alpha + \beta$. When $Z^\vee$ is defined by rotation and the respective edge is taken accordingly, the odd edged part of $Z^\vee$, α', will be equivalent to α because they will have the same number of edges and the same orientation. The position created from $Z^\vee$ will be $\alpha' + \beta'$, with β' being $\beta^\vee$. Therefore, $\alpha' + \beta'$ is equivalent to $\alpha + \beta^\vee$. Since β has an even amount of edges, β and $\beta^\vee$ will be equivalent, making $\alpha + \beta$ and $\alpha' + \beta'$ equivalent. This further proves that Z and $Z^\vee$ are equivalent if and only if they are even edged, as any move on Z will give the same position as if it were made on $Z^\vee$ using the rotation definition. If the edge is taken from the end of Z and from $Z^\vee$, the resulting odd edged zig-zags will still be positioned the same and will be equivalent. □

Theorem 2.19. *There are three cases, some of which contain sub cases, that dictate the winning strategy in an impartial game of Roads that takes the form of a zig-zag.*

1. ***Case 1:*** *Given a zig-zag with* $2(2n) + 2$ *edges, the winning strategy is to remove the middlemost two edges.*
2. ***Case 2:*** *Given a zig-zag with* $2(2n) + 1$ *edges, the winning strategy is to remove the middlemost edge, if the 1st player can do so.*

(3) ***Case 3:*** *Given a zig-zag with* $2(2n+1)+1$ *edges, the winning strategy is to take the middlemost edge if the 1st player can do so.*

Proof. This proof will go through all three cases in Theorem 2.18.

Case 1: Given a zig-zag with $2(2n)+2$ edges, the winning strategy is to remove the middlemost two edges. This provides the next player, Roberto if Richard goes first and vice versus, with two even-edged, equivalent zig-zags, which by Lemma 1.1 sum to type P, giving the 1st player the winning strategy.

Case 2: Given a zig-zag with $2(2n)+1$ edges, the winning strategy is to remove the middlemost edge, if the 1st player can do so. The case in which the 1st player will not be able to remove the middlemost edge, will be when the zig-zag is type X. A type X zig-zag in Impartial Roads is one where the middlemost edge has a positive slope, forcing either Richard or Roberto to take at least two edges if they act on it.

In the case that the zig-zag is not type X, the second player is left with two even edged zig-zags, one Z, and the other $Z^\vee$ based on the definitions in Lemma 2.18. By Lemma 2.18, Z and $Z^\vee$ are equivalent, so their sum, by Lemma 2.17, is type P. As the 1st player moves to a position of type P by taking the middlemost edge, they have a winning strategy.

Case 3: Given a zig-zag with $2(2n+1)+1$ edges, the winning strategy is to take the middlemost edge if the 1st player can do so. If the zig-zag is type X, the winning strategy is unknown and the game may be of type P.

In the case that the zig-zag is not type X, the second player is left with the sum of two equivalent, odd-edged zig-zags. This sum, by Lemma 2.17 is type P, giving the 1st player a winning strategy. □

Remark 2.20. Given a zig-zag with $2(2n+1)+2$ edges, the winning strategy is unclear. Taking the middle two edges will result in two odd edged zig-zags that are not equivalent because they are each others' flips (in other words, Z and $Z^\vee$) by Lemma 2.18.

Example 2.21. To illustrate Remark 2.20, consider the case $n=1$, in which case we have a zig-zag with $2(2+1)+2=8$ edges (see Figure 3). Removing the two middle edges yields

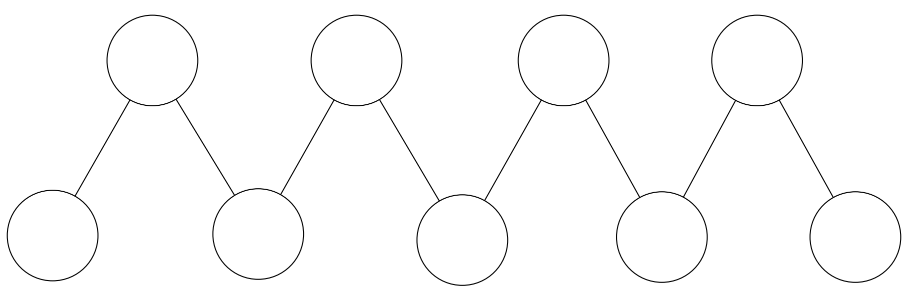


FIGURE 3. A zig-zag with eight edges.

the position shown in Figure 4; this position is of the form $Z+Z^\vee$, where Z has three edges. Therefore, Lemma 2.18 gives that Z and $Z^\vee$ are not equivalent. It follows that the position in Figure 4 is not necessarily type P, making the winning strategy from the position in Figure 3 unclear.

2.4.2. *Partizan Version.*

Lemma 2.22. *The position* $Z+Z^\vee$*, where* $Z^\vee$ *is defined as the flip of* Z *by process of vertical flipping, is type* P*.*

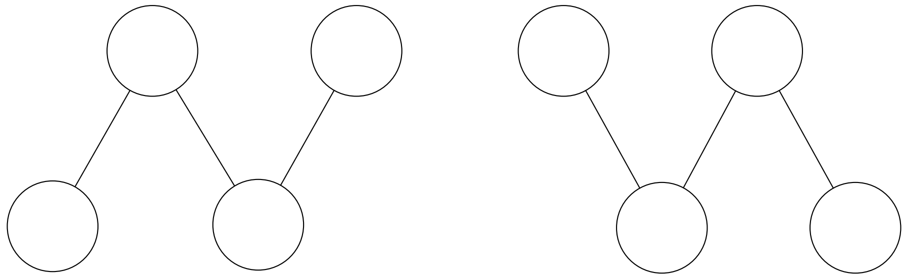


FIGURE 4. A position of the form $Z + Z^\vee$, where Z has three edges.

Proof. Suppose that Z is some zig-zag with n edges in a Partizan game of Roads, and $Z^\vee$ is the position resulting from it being flipped vertically.

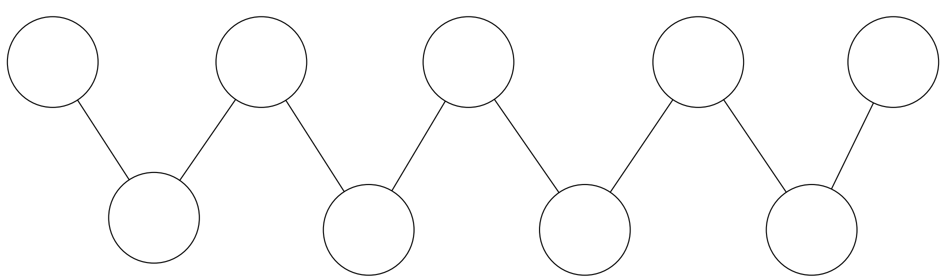
If $Z^\vee$ is defined as the vertical flip of Z, any move on Z can be translated onto $Z^\vee$. As both players, Louise and Richard can take two edges the same way in a zig-zag, we only need to look at the translation of a move that takes one edge.

Richard can only take negative sloped single edges without taking anymore, and Louise can only take positive sloped single edges without taking anymore. Suppose we have a zig-zag, Z, that contains at least one negatively sloped edge. In $Z^\vee$, the vertical flip of Z, this edge would be positive sloped. Therefore, what Richard can take on Z, Louise can take on $Z^\vee$. In the case that Z has no negative sloped single edges, the opposite is true: what Louise can take on Z, Richard can take on $Z^\vee$.

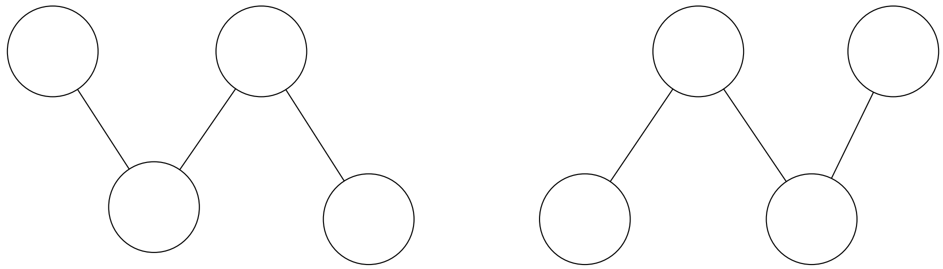
Using an argument of symmetry, it is found that in the Partizan game of Roads, $Z + Z^\vee$ is type P. □

Theorem 2.23. *If there is an even number of roads in a zig-zag, the first player can win by taking the two middle edges.*

Proof. Suppose we have a zig-zag with $2n$ edges.

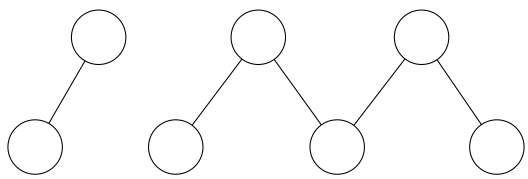


If we were to take away the two middle edges of this zig-zag, we would be left with two zig-zags with n edges, where one zig-zag is denoted as Z and the other is denoted as $Z^\vee$. In this case, $Z^\vee$ is defined as Z flipped horizontally. As we established earlier, and $Z + Z^\vee$ is a type P, meaning that the original zig-zag is type N.



□

2.5. **Conjectures.** This subsection will outline a few patterns which we have noticed within the games of Impartial and Partizan Roads, but which we have not been able to make formal theorems and proofs of. We will also outline further research that may be done on the games of Impartial and Partizan Roads.



The above figure is the sum of a single edged zig-zag and a four edged zig-zag. Evaluating this position in Impartial Roads finds it to be type P. Other similar positions of type P are the sum of a single edged zig-zag and a seven edged zig-zag, as well as the sum of two single edged zig-zags.

Furthermore, the evaluation of the sums of zig-zags within the Impartial game of Roads would benefit the classification of singular zig-zags into different types. It may resolve some of the exceptions to the four cases detailed in Impartial Roads.

For zig-zags in Partizan Roads, even edged zig-zags are proven to be type N. In addition, there is an abundance of odd edged zig-zags which are also type N. This often results from one player being able to take the middlemost single edge, resulting in the sum of two even edged zig-zags, which, by Lemma 2.18, is a position of type P. Meanwhile, the other player can often move to a sum position of their type, being L if they are Louise and R if they are Richard. Since each player, if they move first, has a winning strategy, these odd edged zig-zags are often type N.

Finding the sums of zig-zags within Partizan Roads which are of type L, R, or P would allow greater classification of singular zig-zags within the game. Sums with $2(2n)+1$ edges will result in a type P if a player can remove the middlemost edge because this will give a position of $Z + Z^{\vee}$, which is type P. However, if one player can take the middlemost edge, the other cannot in Partizan Roads. Because of this, it requires further exploration to determine which zig-zags with $2(2n)+1$ edges are which type. Furthermore, zig-zags with $2(2n+1)+1$ edges are also of an undetermined type because removing the middlemost edge doesn't necessarily result in a position of type P, and if one player can do this, the other cannot.

A next step in analyzing Roads would be to connect its two versions: Impartial Roads and Partizan Roads. While theorems and proofs may not be easily transferable from one of these games to another, Impartial Roads essentially deals with a game between Richard and Richard. If nothing else, the analysis of Impartial Roads shows what moves Richard may be able to make in Partizan Roads from a given position.

3. Classical Game Theory

3.1. **Basic Definitions.** This section overviews the basic definitions that pertain to Classical Game Theory and which will help us analyze our original game, One Four All.

Definition 3.1 ([1, Pg.89-90])**.** *Classical game theory* is the study of games in which both players simultaneously, without knowledge of the other player's decision, choose a course of action, resulting in a single outcome.

Remark 3.2. In classical game theory there are two players Rose and Colin. Rose can choose a row and Colin can choose a column in a matrix representing a classical game.

Definition 3.3 ([1, Pg. 90])**.** A *zero-sum matrix game* is a game played between Rose and Colin. There is a fixed matrix that is known to both Rose and Colin they secretly choose a row or a column (Rose will choose a row and Colin will choose a column). Given Rose chooses some row, i, and Colin chooses some column j, in some matrix A, the *payoff* is the (i, j) entry of matrix A. Rose wants the highest possible payoff and Colin wants the lowest possible payoff.

Definition 3.4 ([1, Pg. 92])**.** *Dominance* is used in order to simplify a zero-sum matrix game. When given a zero-sum matrix game, we can use dominance by eliminating the rows with the lowest payoff and the columns with the highest payoff. We can do this because Rose always wants the highest payoff so a row in which each value in the row is smaller then a corresponding value in another of the matrix is no use to Rose as it will never be beneficial for Rose to chose that row. The opposite goes for Colin who always wants the lowest payoff.

Definition 3.5 ([1, Pg. 90])**.** A *mixed strategy* is a strategy in which Rose (resp. Colin) picks row (resp. column) with certain probabilities. All the entries of a mixed strategy sum to one.

Definition 3.6 ([1, Pg. 100])**.** For a given mixed strategy p, which Rose will play, the *guarantee* is the minimum entry pA, with A as some matrix. For Colin, the guarantee is the highest entry of qA given a mixed strategy q, and a matrix, A.

Throughout this section we will make use of the following theorem which will describe how to solve a 2×2 matrix.

Theorem 3.7 ([1, Pg. 111])**.** *Von Nuemann Minimiax Theorem: Every zero-sum matrix game has a value v and pair mixed strategies $\mathbf{p}$ for Rose and $\mathbf{q}$ for Colin so that both $\mathbf{p}$ and $\mathbf{q}$ have a guarantee of v.*

The value v, together with $\mathbf{p}$ and $\mathbf{q}$, is called a *Von Neumann solution* to the zero-sum matrix game.

Proposition 3.8 ([1, Proposition 6.4])**.** *For every 2×2 zero-sum matrix game A, one of the following holds.*

(1) Iterated removal of dominated strategies reduces the matrix to a 1×1 matrix $[v]$. The number v and the associated pure row and column strategies form a Von Neumann Solution.

(2) Rose and Colin have mixed strategies $\mathbf{p}$ and $\mathbf{q}$ equating the opponents' results. Then

$$\mathbf{p}A = \begin{bmatrix} v & v \end{bmatrix}$$

and

$$A\mathbf{q} = \begin{bmatrix} v \\ v \end{bmatrix} .$$

The number v, together with $\mathbf{p}$ *and* $\mathbf{q}$*, form a Von Neumann solution.*

3.2. **One Four All.** We first describe the rules for our game, One Four All. Our game starts with a 2×2 matrix containing the values 1, 2, 3, and 4 that looks like this:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} .$$

Rose has a chip valued at $+1$ and Colin has a chip valued at -1. They each place a there chip down at the same time. The score is calculated by taking the the value of the chip, multiplying it by its corresponding matrix value and then adding the two together. However, if Rose and Colin choose to put their chips on 4 and 1, then the score is -3. By going through our game and looking at possible outcomes, we get the following payoff matrix:

$$\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ -3 & 2 & 1 & 0 \end{bmatrix} .$$

As you can see, there is some dominance in this matrix that will allow for it to be simplified. After using dominance, we get the following 2×2 matrix:

$$\begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix} .$$

The next step in solving this zero-sum matrix game is to use Von Neumann's Minimax Theorem. In order to do that, we are going to multiply our matrix first on the left by a 1×2 matrix with the entries p and $1 - p$; we will then solve for p by setting the entries in the resulting matrix equal to each other. Next, we will then repeat this process but now solving for Colin's q value by multiplying our matrix on the right by a 2×1 matrix with the entries q and $1 - q$. We will then plug p and q into their respective matrices and multiply everything out. This will give us our guarantees for Colin and for Rose, which we expect to be equal to the same value v (by Proposition 3.8). Then v, p, and q define a Von Neumann solution to One Four All.

We begin this process as follows:

$$\begin{bmatrix} p & 1-p \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 5p-3 & -p \end{bmatrix} .$$

By setting the two entries equal, we get $5p - 3 = -p$; therefore, $p = 1/2$. Repeating the process with q yields the following:

$$\begin{bmatrix} q \\ 1-q \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 3q-1 \\ -3q \end{bmatrix} .$$

Setting the two entries of the resulting matrix equal yields $3q - 1 = -3q$; therefore, $q = 1/6$. In a final step we plug p and q back in. By plugging p back in we get the following:

$$\begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \end{bmatrix} .$$

The entries in the resulting matrix are equal; therefore, Rose has a guarantee of $-1/2$. Now by plugging q back in we get the following:

$$\begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix}.$$

The two entries in the matrix are again the same; therefore, Colin has a guarantee of $-1/2$.

Conclusion: Both Rose and Colin have a guarantee of $-1/2$. That means that in our matrix game, Colin has the winning strategy, because no matter what Rose does Colin is guaranteed $-1/2$.

4. Acknowledgements

We would like to thank everyone at PRIMES Circle, including Peter Haine, who has put so much work and effort into making this program possible. We would also like to thank our mentor Marisa, who guided us through this entire program making it possible for us to write this paper.

References

[1] M. DeVos and D. A. Kent. *Game Theory: A Playful Introduction. American Mathematical Society* Vol. 80, 2016.